Problem 1.

Find the critical points of the function

$$f(x, y) = x^3 + 6xy + 3y^2 - 9x$$

and determine their nature.

We calculate

$$f_x = 3x^2 + 6y - 9, \; f_y = 6x + 6y.$$

Setting

$$f_y = 0 \implies y = -x.$$

Using $f_x = 0$ we find

$$3x^2 - 6x - 9 = 0 \implies x^2 - 2x - 3 = 0 \implies x = -1 \text{ or } x = 3.$$

We find the critical points $(-1, 1)$ and $(3, -3)$. To find the nature of the critical points we use the second derivative test. We have

$$f_{xx} = 6x, f_{yy} = 6, f_{xy} = 6.$$

At the point $(-1, 1)$, we have $H_f = \begin{bmatrix} -6 & 6 \\ 6 & 6 \end{bmatrix}$ which has negative determinant hence

$$\boxed{(-1, 1) \text{ is a saddle point}}.$$

At the point $(3, -3)$, we have $H_f = \begin{bmatrix} 18 & 6 \\ 6 & 6 \end{bmatrix}$ which has negative determinant hence

$$\boxed{(3, -3) \text{ is a local minimum}}.$$

Problem 2.

Find the minimum and the maximum of the function

$$f(x, y) = x^2 + 2y^2 - 6x + 2$$

along the ellipse

$$2x^2 + y^2 = 8.$$

Write $g(x, y) = 2x^2 + y^2$. We have $\nabla g = (4x, 2y) \neq (0, 0)$ when $g = 8$. Hence

$$\nabla f = \lambda \nabla g \implies (2x - 6, 4y) = \lambda(4x, 2y).$$

The second equation gives

$$4y = 2\lambda y$$

hence either $y = 0$ or $\lambda = 2$. When $\lambda = 2$, we obtain

$$2x - 6 = 4\lambda x = 8x \implies x = -1 \implies y^2 = 6 \implies f(-1, \pm\sqrt{6}) = 21.$$

When $y = 0$ we obtain

$$2x^2 = 8 \implies x = \pm 2 \implies f(2, 0) = -6, f(-2, 0) = 18.$$

Thus $(-1, \pm\sqrt{6})$ gives the maximum value $\boxed{21}$ and $(2, 0)$ gives the minimum value $\boxed{-6}$.

Problem 3.

Assume that $z = f(x, y)$ and

$$x = \frac{u^2}{v},\ y = 2uv - v^2.$$

We are given

$$f_x(4, 3) = 2,\ \frac{\partial f}{\partial y} = x - 2y.$$

Calculate

$$\frac{\partial z}{\partial v}$$

at the point $u = 2$ and $v = 1$.

Note that when $u = 2, v = 1$ we must have $x = 4$ and $y = 3$. Hence

$$\frac{\partial z}{\partial x} = 2 \text{ and } \frac{\partial z}{\partial y} = x - 2y \implies \frac{\partial z}{\partial y} = -2 \text{ at } x = 4, y = 3.$$

Now

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

We have

$$\frac{\partial x}{\partial v} = -\frac{u^2}{v^2} \implies \frac{\partial x}{\partial v} = -4 \text{ at } u = 2, v = 1.$$

Finally,

$$\frac{\partial y}{\partial v} = 2u - 2v \implies \frac{\partial y}{\partial v} = 2 \text{ at } u = 2, v = 1.$$

Therefore

$$\frac{\partial z}{\partial v} = 2 \cdot (-4) + (-2) \cdot 2 = \boxed{-12}.$$

Problem 4.

Find the volume of the region bounded on the top by the paraboloid $z = 4 - x^2 - 3y^2$, on the bottom by the (x, y)-plane, on the sides by the planes $x = 0$, $x = 1$, $y = -1$ and $y = 1$.

We find

$$\int_0^1 \int_{-1}^1 4 - x^2 - 3y^2 \; dy \, dx.$$

The inner integral equals

$$\int_{-1}^1 4 - x^2 - 3y^2 \, dy = 4y - x^2 y - y^3 |_{-1}^1 = 8 - 2x^2 - 2 = 6 - 2x^2.$$

The outer integral equals

$$\int_0^1 6 - 2x^2 \, dx = 6x - \frac{2}{3} x^3 |_0^1 = 6 - \frac{2}{3} = \boxed{\frac{16}{3}}.$$

Problem 5.

Consider the function

$$f(x, y) = x^2y^4 + xy^2 \ln(2x - y).$$

(i) Find the unit direction of steepest increase for the function f at the point P.

We have

$$f_x = 2xy^4 + y^2 \ln(2x - y) + xy^2 \cdot \frac{2}{2x - y} \implies f_x(1, 1) = 4.$$

Next,

$$f_y = 4x^2y^3 + 2xy \ln(2x - y) + xy^2 \cdot \frac{-1}{2x - y} \implies f_y(1, 1) = 3.$$

Thus $\nabla f(1, 1) = (4, 3)$. The direction of steepest increase is $(4, 3)$.

(ii) Calculate the directional derivative $D_{\vec{v}}f(P)$ in the direction $\vec{v} = \frac{1}{\sqrt{2}}\vec{\imath} - \frac{1}{\sqrt{2}}\vec{\jmath}$,

We have

$$D_{\vec{v}}f(P) = \nabla f \cdot \vec{v} = (4, 3) \cdot (1/\sqrt{2}, -1/\sqrt{2}) = \frac{1}{\sqrt{2}}.$$

(iii) Calculate the tangent plane to the graph of f at the point $(P, f(P))$.

We have $f(P) = 1$ hence

$$z - 1 = 4(x - 1) + 3(y - 1) \implies z = 4x + 3y - 6.$$

(iv) Find the tangent plane to the surface $z^2x^3 - f(x, y) = 0$ at the point $(1, 1, 1)$.

We set $g(x, y, z) = z^2x^3 - f(x, y)$. We find

$$g_x = 3x^2z^2 - f_x \implies g_x(1, 1, 1) = 3 - 4 = -1$$
$$g_y = -f_y \implies g_y(1, 1, 1) = -3$$
$$g_z = 2zx^3 \implies g_z(1, 1, 1) = 2.$$

The tangent plane is

$$-(x - 1) - 3(y - 1) + 2(z - 1) = 0 \implies -x - 3y + 2z = -2.$$