PRACTICE PROBLEMS FOR THE FINAL EXAM

Problem 1.

At what point (x, y, z) on the plane $x + 2y - z = 6$ does the minimum of the function

$$f(x, y, z) = (x-1)^2 + 2y^2 + (z+1)^2$$

occur?

Problem 2.

Consider the function

$$f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy.$$

(i) Find the critical points of the function.
(ii) Determine the nature of the critical points (local min/local max/saddle).
(iii) Does the function $f(x, y)$ have a global minimum or a global maximum?

Problem 3.

Consider the function

$$f(x, y) = \ln(xy^2) - \frac{2x}{y}$$

(i) Find the tangent plane to the graph of f at the point $(1, 1, -2)$.
(ii) Estimate the value of $f(1.01, .99)$.
(iii) Find the tangent plane to the surface S:

$$z^2xy^3 - zf(x^2, y^3) = 3$$

at the point $(1, 1, 1)$.

Problem 4.

Find the global minimum and global maximum of the function

$$f(x, y) = x^2 + y^2 - 2x - 2y + 4$$

over the closed disk

$$x^2 + y^2 \leq 8.$$

Problem 5.

Consider the function

$$f(x, y) = 1 + \sqrt{x^2 + y^2}.$$

(i) Draw the contour diagram of f labeling at least three levels of your choice.
(ii) Compute the gradient of f at $(1, -1)$ and draw it on the contour diagram of part (i).
(iii) Does the function f have a global minimum? If no, why not? If yes, what is the minimum value?
(iv) Draw the graph of the function f.

Problem 6.

Consider the function
$$f(x,y) = e^{-3x+2y}\sqrt{2x+1}\,.$$

(i) Calculate the gradient of f at $(0,0)$.
(ii) Find the directional derivative of f at $(0,0)$ in the direction $\mathbf{u} = \frac{i+j}{\sqrt{2}}$.
(iii) What is the unit direction for which the rate of increase of f at $(0,0)$ is maximal?

Problem 7.

Consider the planes
$$x + 2y - z = 1,\ x + 4y - 2z = 3.$$

(i) Find normal vectors to the two planes.
(ii) Are the two planes parallel? Are they perpendicular? What is the angle between the planes?
(iii) Find the parametrization of the line of intersection of the two planes.
(iv) Find a third plane parallel to the intersection line you found in (iii), which passes through the points $P(1,0,1)$ and $Q(-1,2,1)$.

Problem 8.

Consider the function
$$w = u^2 v\, e^{-v}$$
and assume that
$$u = x^2 - 2xy,\ v = -x + 2\ln y.$$
Calculate the values of the derivatives
$$\frac{\partial w}{\partial x} \text{ and } \frac{\partial w}{\partial y}$$
at the point $(x,y) = (1,1)$.

Problem 9.

Determine the average value of the function
$$f(x,y,z) = \sqrt{x^2+y^2+z^2}\, e^{x^2+y^2+z^2}$$
over the region D bounded by the two spheres $x^2+y^2+z^2 = a^2$ and $x^2+y^2+z^2 = b^2$, for $0 < a < b$.

Problem 10.

Find the total mass of the region W that represents the intersection of the solid cylinder $x^2+y^2 \leq 1$ and the solid ellipsoid $2(x^2+y^2)+z^2 \leq 10$ given that the density $\delta = 1$.

Problem 11.

Find the centroid of the region bounded above by the sphere of radius 5 and below by the cone $z = 2\sqrt{x^2+y^2}$

Problem 12.

Find the volume of the region in the first octant that lies inside the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 - 2x = 0$

Problem 13.

Evaluate

$$\int_0^2 \int_{\frac{y}{2}}^1 e^{-x^2}\, dxdy.$$

Problem 14.

Calculate the limits below or explain why they do not exist

(i) $\lim_{x,y,z\to 0} \frac{x^2y^2z^2}{x^4+y^4+z^4}$.
(ii) $\lim_{x,y\to 0} \frac{xy^2}{x^2+4y^4}$.

Problem 15.

Evaluate $\iint_D x^2 + y^2 dA$ where D is the region in the first quadrant bounded by

$$y = 3x,\ y = x,\ xy = 3.$$

Problem 16.

Find the arclength parametrization of the cycloid

$$x = t - \sin t,\ \ y = 1 - \cos t,\ \ 0 \leq t \leq 2\pi.$$

What is the length of the cycloid?

Problem 17.

Calculate the volume of the region bounded by the parabolic cylinder $x = y^2$, the planes $z = 0$ and $x + z = 1$.

Problem 18.

Find the integral

$$\int\int\int_D |y - 1|\, dV$$

where D is the oblique segment of a paraboloid bounded by $z = x^2 + y^2$ and the plane $z = 2y + 3$.