Problem 1.

At what point (x, y, z) on the plane $x + 2y - z = 6$ does the minimum of the function

$$f(x, y, z) = (x - 1)^2 + 2y^2 + (z + 1)^2$$

occur?

`Answer:` Using Lagrange multipliers, we need to solve

$$\nabla f = \lambda \nabla g$$

where

$$g(x, y, z) = x + 2y - z.$$

Computing the gradients, we conclude

$$(2x - 2, 4y, 2(z + 1)) = \lambda(1, 2, -1) \implies x = \frac{\lambda}{2} + 1, y = \frac{\lambda}{2}, z = -\frac{\lambda}{2} - 1.$$

Substituting, we have

$$x + 2y - z = 4 \implies \frac{\lambda}{2} + 1 + 2 \cdot \frac{\lambda}{2} + \frac{\lambda}{2} + 1 = 6 \implies \lambda = 2.$$

This gives

$$x = 2, y = 1, z = -2.$$

Problem 2.

Consider the function

$$f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy.$$

(i) *Find the critical points of the function.*
(ii) *Determine the nature of the critical points (local min/local max/saddle).*
(iii) *Does the function $f(x, y)$ have a global minimum or a global maximum?*

`Answer:`

(i) We have

$$f_x = -6x + 6y = 0 \implies x = y$$
$$f_y = 6y - 6y^2 + 6x = 0 \implies y - y^2 + x = 0.$$

Substituting $x = y$ into the second equation, we obtain

$$2y - y^2 = 0 \implies y = 0 \text{ or } y = 2.$$

The critical points are

$$(0, 0), (2, 2).$$

(ii) We compute

$$A = f_{xx} = -6, \;\; B = f_{xy} = 6, \;\; f_{yy} = 6 - 12y.$$

When

$$x = y = 0 \implies AC - B^2 = (-6)(6) - 6^2 < 0 \implies (0, 0) \text{ saddle point},$$
$$x = y = 2 \implies AC - B^2 = (-6)(-18) - 6^2 > 0, A < 0 \implies (2, 2) \text{ local max.}$$

(iii) We compute

$$\lim_{x\to\infty, y=0} f(x,y) = \lim_{x\to\infty} -3x^2 = -\infty \implies \text{no global min.}$$

Similarly,

$$\lim_{x=0, y\to-\infty} f(x,y) = \lim_{y\to-\infty} 3y^2 - 2y^3 = \infty \implies \text{no global max.}$$

Problem 3.

Consider the function

$$f(x,y) = \ln(xy^2) - \frac{2x}{y}$$

(i) *Find the tangent plane to the graph of f at the point $(1,1,-2)$.*
(ii) *Estimate the value of $f(1.01, .99)$.*
(iii) *Find the tangent plane to the surface S:*

$$z^2xy^3 - zf(x^2, y^3) = 3$$

at the point $(1,1,1)$.

`Answer:`

(i) We compute

$$f(1,1) = \ln 1 - 2 = -2$$

$$f_x(x,y) = \frac{y^2}{xy^2} - \frac{2}{y} = \frac{1}{x} - \frac{2}{y} \implies f_x(1,1) = -1$$

$$f_y(x,y) = \frac{2xy}{xy^2} + \frac{2x}{y^2} = \frac{2}{y} + \frac{2x}{y^2} \implies f_y(1,1) = 4$$

The tangent plane is

$$z + 2 = -(x-1) + 4(y-1) \implies z = -x + 4y - 5.$$

(ii) We have

$$z + 2 \approx -(1.01 - 1) + 4(.99 - 1) = -.05 \implies z = f(1.01, .99) \approx -2.05.$$

(iii) Write

$$g(x,y,z) = z^2xy^3 - zf(x^2, y^3).$$

We calculate

$$g_x = z^2y^3 - 2zxf_x(x^2, y^3) \implies g_x(1,1,1) = 1 - 2f_x(1,1) = 3$$

$$g_y = 3z^2xy^2 - 3zy^2f_y(x^2, y^3) \implies g_y(1,1,1) = 3 - 3f_y(1,1) = -9$$

and

$$g_z = 2zxy^3 - f(x^2, y^3) \implies g_z(1,1,1) = 2 - f(1,1) = 4.$$

The tangent plane has normal vector $(3, -8, 4)$ and equation

$$3(x-1) - 9(y-1) + 4(z-1) = 0 \implies 3x - 9y + 4z = -2.$$

Problem 4.

Find the global minimum and global maximum of the function

$$f(x, y) = x^2 + y^2 - 2x - 2y + 4$$

over the closed disk

$$x^2 + y^2 \leq 8.$$

`Answer:`

We find the critical points in the interior by setting the partial derivatives to zero:

$$f_x = 2x - 2 = 0 \implies x = 1$$
$$f_y = 2y - 2 = 0 \implies y = 1.$$

We get the critical point $(1, 1)$ with value

$$f(1, 1) = 2.$$

We check the boundary $g(x, y) = x^2 + y^2 = 8$ using Lagrange muliplier*s*

$$\nabla f = \lambda \nabla g \implies (2x - 2, 2y - 2) = \lambda(2x, 2y) \implies x - 1 = \lambda x, y - 1 = \lambda y.$$

Dividing we obtain

$$\frac{x-1}{y-1} = \frac{\lambda x}{\lambda y} = \frac{x}{y} \implies y(x - 1) = x(y - 1) \implies xy - y = xy - x \implies x = y.$$

Since

$$x^2 + y^2 = 8 \implies x = y = \pm 2.$$

We evaluate

$$f(2, 2) = 4, \ f(-2, -2) = 20.$$

Therefore $(1, 1)$ is the global minimum, while $(-2, -2)$ is the global maximum.

Problem 5.

Consider the function

$$f(x, y) = 1 + \sqrt{x^2 + y^2}.$$

(i) *Draw the contour diagram of f labeling at least three levels of your choice.*
(ii) *Compute the gradient of f at $(1, -1)$ and draw it on the contour diagram of part (i).*
(iii) *Does the function f have a global minimum? If no, why not? If yes, what is the minimum value?*
(iv) *Draw the graph of the function f.*

`Answer:`

(i) The contour diagram consists of circles centered at the origin. Indeed,

$$f(x,y) = c \implies x^2 + y^2 = (c-1)^2.$$

Level c corresponds to a circle of radius $c-1$.

You may use three values for c to draw the contour diagram. For instance, for $c = 2, 3, 4$, we get circles of radii $1, 2, 3$ respectively.

(ii) The gradient is

$$\nabla f = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right) \implies \nabla f(1,-1) = \frac{1}{\sqrt{2}}(1,-1).$$

This vector is normal to the level curve.

(iii) The global minimum occurs at $(0,0)$. The minimum value is $f(0,0) = 1$.

(iv) The graph is a cone with vertex $(0,0,1)$.

Problem 6.

Consider the function

$$f(x,y) = e^{-3x+2y}\sqrt{2x+1}\,.$$

(i) *Calculate the gradient of f at $(0,0)$.*

(ii) *Find the directional derivative of f at $(0,0)$ in the direction $\mathbf{u} = \frac{i+j}{\sqrt{2}}$.*

(iii) *What is the unit direction for which the rate of increase of f at $(0,0)$ is maximal?*

`Answer:`

(i) Using the product rule and the chain rule, we have

$$f_x = -3e^{-3x+2y}\sqrt{2x+1} + e^{-3x+2y} \cdot \frac{1}{2\sqrt{2x+1}} \cdot 2 \implies f_x(0,0) = -3+1 = -2,$$

$$f_y = 2e^{-3x+2y}\sqrt{2x+1} \implies f_y(0,0) = 2.$$

The gradient is

$$\nabla f = (-2, 2).$$

(ii) We compute

$$f_{\mathbf{u}} = \nabla f \cdot \mathbf{u} = (-2,2) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = 0.$$

(iii) The direction is parallel to the gradient, normalized to have unit length

$$\mathbf{v} = \frac{\nabla f}{||\nabla f||} = \frac{(-2,2)}{\sqrt{(-2)^2+2^2}} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right).$$

Problem 7.

Consider the planes

$$x + 2y - z = 1,\ x + 4y - 2z = 3.$$

(i) *Find normal vectors to the two planes.*

(ii) *Are the two planes parallel? Are they perpendicular? What is the angle between the planes?*

(iii) *Find the parametrization of the line of intersection of the two planes.*

(iv) *Find a third plane parallel to the intersection line you found in (iii), which passes through the points $P(1,0,1)$ and $Q(-1,2,1)$.*

`Answer:`

(i) The normal vectors are given by the coefficients of the two planes

$$\vec{n}_1 = (1,2,-1),\ \vec{n}_2 = (1,4,-2).$$

(ii) The vectors $\vec{n}_1, \vec{n}_2$ are not proportional hence the planes are not parallel. We have

$$\vec{n}_1 \cdot \vec{n}_2 = (1,2,-1)\cdot(1,4,-2) = 1+8+2 \neq 0.$$

The planes are not perpendicular. In fact, we can calculate the angle between the normal vectors as

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{||\vec{n}_1||\cdot||\vec{n}_2||} = \frac{11}{\sqrt{6}\cdot\sqrt{21}}.$$

(iii) The line of intersection is normal to both n_1 and n_2 hence it is parallel to the cross product

$$\vec{n}_1 \times \vec{n}_2 = (0,1,2).$$

A point of intersection can be found by setting $z=1$. We find $x+2y=2$ and $x+4y=5$ hence $y=3/2$ and $x=-1$. The line of intersection is

$$(-1,3/2,1)+t(0,1,2).$$

(iv) We have

$$\vec{PQ} = (-2,2,0).$$

The plane is parallel to the vectors $(-2,2,0)$ and $(0,1,2)$. A normal vector is

$$\vec{n} = (-2,2,0)\times(0,1,2) = (4,4,-2).$$

The plane is

$$4x+4y-2z=2$$

using that it passes through either P or Q.

Problem 8.

Consider the function

$$w = u^2 v\, e^{-v}$$

and assume that

$$u = x^2 - 2xy,\ v = -x + 2\ln y.$$

Calculate the values of the derivatives

$$\frac{\partial w}{\partial x} \text{ and } \frac{\partial w}{\partial y}$$

at the point $(x,y)=(1,1)$.

`Answer:` We evaluate

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x}.$$

At $x=y=1$, we have

$$u=-1, v=-1.$$

We compute

$$\frac{\partial w}{\partial u} = 2uve^{-v} = 2e$$

$$\frac{\partial w}{\partial v} = u^2 e^{-v} - u^2 v e^{-v} = e + e = 2e.$$

Furthermore,

$$\frac{\partial u}{\partial x} = 2x - 2y = 0$$
$$\frac{\partial v}{\partial x} = -1.$$

Thus

$$\frac{\partial w}{\partial x} = 2e \cdot 0 + 2e \cdot (-1) = -2e.$$

A similar calculation shows

$$\frac{\partial w}{\partial y} = 0.$$

Problem 9.

Determine the average value of the function

$$f(x,y,z) = \sqrt{x^2+y^2+z^2}\, e^{x^2+y^2+z^2}$$

over the region D *bounded by the two spheres* $x^2+y^2+z^2 = a^2$ *and* $x^2+y^2+z^2 = b^2$, *for* $0 < a < b$.

`Answer:` Spherical coordinates are most convenient. The region is defined by $a \le \rho \le b$. We have

$$\text{Volume}(D) = \frac{4\pi}{3}(b^3 - a^3).$$

Next,

$$\int_D f\,dV = \int_0^{2\pi}\int_0^{\pi}\int_a^b \rho e^{\rho^2}\rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta = \left(\int_0^{2\pi}\int_0^{\pi} \sin\varphi\,d\varphi\,d\theta\right)\int_a^b \rho^3 e^{\rho^2}\,d\rho = 4\pi\int_a^b \rho^3 e^{\rho^2}\,d\rho$$

The integrand is equal to $u\,dv$, where $u = \rho^2$, $dv = \rho e^{\rho^2}\,d\rho$, and $v = \frac{1}{2}e^{\rho^2}$. We apply integration by parts:

$$= 4\pi\Big[uv\Big]_{\rho=a}^{\rho=b} - 4\pi\int_{\rho=a}^{\rho=b} v\,du = 4\pi\left[\frac{1}{2}\rho^2 e^{\rho^2}\right]_a^b - 4\pi\int_a^b \rho e^{\rho^2}\,d\rho$$

$$= 2\pi(b^2e^{b^2} - a^2e^{a^2}) - 2\pi\left[e^{\rho^2}\right]_a^b = 2\pi(b^2e^{b^2} - e^{b^2} - a^2e^{a^2} + e^{a^2}) = 2\pi((b^2-1)e^{b^2} - (a-1)e^{a^2}).$$

To find the average, we divide the above integral by the volume, to find

$$\text{average} = \frac{3((b^2-1)e^{b^2} - (a-1)e^{a^2})}{2(b^3-a^3)}.$$

Problem 10.

Find the total mass of the region W *that represents the intersection of the solid cylinder* $x^2+y^2 \le 1$ *and the solid ellipsoid* $2(x^2+y^2)+z^2 \le 10$ *given that the density* $\delta = 1$.

`Answer:` The cylinder and ellipsoid are symmetric around the z axis, so cylindrical coordinates are convenient. The region is defined by

$$r \le 1, \;\; -\sqrt{10-2r^2} \le z \le \sqrt{10-2r^2}.$$

To find the mass, we integrate the density. We find

$$\text{mass} = \int_0^{2\pi}\int_0^1\int_{-\sqrt{10-2r^2}}^{\sqrt{10-2r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r\sqrt{10-2r^2}\,dr\,d\theta = \int_0^{2\pi}\left[-\frac{1}{3}(10-2r^2)^{3/2}\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi}\frac{1}{3}10^{3/2}-\frac{1}{3}8^{3/2}\,d\theta = \int_0^{2\pi}\frac{1}{3}(10\sqrt{10}-8\sqrt{8})\,d\theta = \frac{2\pi}{3}(10\sqrt{10}-16\sqrt{2}) = \boxed{\frac{4\sqrt{2}\pi}{3}(5\sqrt{5}-8)}.$$

Problem 11.

Find the centroid of the region bounded above by the sphere of radius 5 *and below by the cone* $z = 2\sqrt{x^2+y^2}$

`Answer:`

The region is symmetric about the z-axis, so that $\hat{x}$ and $\hat{y}$ lie on that axis (i.e. both are zero). Thus we need only find

$$\hat{z} = \frac{1}{Vol(W)}\iiint z\,dV.$$

In spherical coordinates

$$= \int_0^{2\pi}\int_0^{\tan^{-1}\frac{1}{2}}\int_0^5 \rho^3\cos\phi\sin\phi\;d\rho d\phi d\theta = 2\pi\int_0^{\tan^{-1}\frac{1}{2}}\left[\frac{\rho^4}{4}\cos\phi\sin\phi\right]_0^5 d\phi$$

$$= \frac{625\pi}{2}\int_0^{\tan^{-1}\frac{1}{2}}\cos\phi\sin\phi\,d\phi = \frac{625\pi}{4}\left[\sin^2\phi\right]_0^{\tan^{-1}\frac{1}{2}}$$

$$= \frac{125\pi}{4}$$

We calculate the volume

$$\text{Vol}(W) = \int_0^{2\pi}\int_0^{\tan^{-1/2}}\int_0^5 \rho^2\sin\phi d\rho\,d\phi\,d\theta = 2\pi\cdot\frac{125}{3}\cdot(1-\frac{2}{\sqrt{5}}).$$

Dividing, we obtain the value for $\hat{z}$.

Problem 12.

Find the volume of the region in the first octant that lies inside the sphere $x^2+y^2+z^2=4$ *and the cylinder* $x^2+y^2-2x=0$.

`Answer:` Note that $x^2+y^2-2x=0$ can be rewritten as $(x-1)^2+y^2=1$. In cylindrical coordinates, the integral is

$$\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}\int_0^{\sqrt{4-r^2}} dzr\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} r\sqrt{4-r^2}\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\frac{-1}{3}\left[(4-r^2)^{3/2}\right]_0^{2\cos\theta} d\theta$$

$$= \frac{8}{3}\int_0^{\frac{\pi}{2}} 1-\sin^3\theta d\theta = \boxed{\frac{8}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)}.$$

Problem 13.

Evaluate

$$\int_0^2 \int_{\frac{y}{2}}^1 e^{-x^2}\, dxdy.$$

`Answer:` First, we change the order of integration:

$$\int_0^2 \int_{\frac{y}{2}}^1 e^{-x^2}\, dxdy = \int_0^1 \int_0^{2x} e^{-x^2}\, dydx = \int_0^1 2xe^{-x^2}\, dx$$

Then, we substitute $u = -x^2 \Rightarrow du = -2xdx$. The integral becomes

$$-\int_0^{-1} e^u du = \int_{-1}^0 e^u du = \Big[e^u\Big]_{-1}^0 = \boxed{\frac{e-1}{e}}.$$

Problem 14.

Calculate the limits below or explain why they do not exist

(i) $\lim_{x,y,z\to 0} \frac{x^2y^2z^2}{x^4+y^4+z^4}$.

(ii) $\lim_{x,y\to 0} \frac{xy^2}{x^2+4y^4}$.

`Answer:`

(i) When $y \le z$, we have

$$y^2z^2 \le z^4 \le x^4 + y^4 + z^4,$$

hence

$$\frac{y^2z^2}{x^4+y^4+z^4} \le 1 \implies 0 \le \frac{x^2y^2z^2}{x^4+y^4+z^4} \le x^2 \to 0.$$

The limit is 0 by the squeeze theorem. When $y > z$, the argument is similar.

(ii) If we let $x, y \to 0$ along the parabola $x = my^2$, the fraction becomes

$$\frac{xy^2}{x^2+4y^4} = \frac{my^4}{m^2y^4+4y^4} = \frac{m}{m^2+4}.$$

This does depend on m, hence the limit does not exist.

Problem 15.

Evaluate $\iint_D x^2 + y^2 dA$ *where* D *is the region in the first quadrant bounded by*

$$y = 3x,\ y = x,\ xy = 3.$$

`Answer:`

$$\iint_D x^2 + y^2\, dA = \int_0^1 \int_x^{3x} x^2 + y^2\, dydx + \int_1^{\sqrt{3}} \int_x^{\frac{3}{x}} x^2 + y^2\, dydx$$

$$= \int_0^1 \left[\frac{y^3}{3} + yx^2\right]_x^{3x} dx + \int_1^{\sqrt{3}} \left[\frac{y^3}{3} + yx^2\right]_x^{\frac{3}{x}} dx$$

$$= \int_0^1 \left(\frac{32x^3}{3}\right)dx + \int_1^{\sqrt{3}} \left(\frac{27+9x^4-4x^3}{3x^3}\right)dx$$
$$= \frac{8}{3}\left[x^4\right]_0^1 + \left[\frac{-27+9x^4-2x^6}{6x^2}\right]_1^{\sqrt{3}} = \boxed{6}.$$

Problem 16.

Find the arclength parametrization of the cycloid

$$x = t - \sin t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi.$$

What is the length of the cycloid?

`Answer:` We evaluate

$$dx = (1-\cos t)\,dt,\ dy = \sin t\,dt$$

hence

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(1-\cos t)^2 + (\sin t)^2}\,dt = \sqrt{2 - 2\cos t}\,dt = 2\sin\frac{t}{2}\,dt$$

using the half angle formula. The length between 0 and t equals

$$s(t) = \int_0^t 2\sin\frac{u}{2}\,du = -4\cos\frac{u}{2}|_{u=0}^{u=t} = 4 - 4\cos\frac{t}{2}.$$

We solve

$$s(t) = s \implies 4 - 4\cos\frac{t}{2} = s \implies \cos\frac{t}{2} = -\frac{s}{4} + 1 \implies t = 2\cos^{-1}\left(1 - \frac{s}{4}\right).$$

The parametric equation becomes

$$x = 2\cos^{-1}\left(1-\frac{s}{4}\right) - \sin\left(2\cos^{-1}\left(1-\frac{s}{4}\right)\right)$$
$$y = 1 - \cos\left(2\cos^{-1}\left(1-\frac{s}{4}\right)\right).$$

Clearly, $t = 0$ corresponds to $s = 0$ and $t = 2\pi$ corresponds to $s = 8$. Thus

$$0 \le s \le 8$$

and the length of the cycloid is therefore 8.

The formulas can be simplified a bit using double angle formulas (This is not necessary for the complete answer, but it's good to remark it nonetheless). For instance using that

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

for

$$\alpha = \cos^{-1}\left(1-\frac{s}{4}\right) \implies \cos\alpha = 1 - \frac{s}{4}$$

we have

$$y = 1 - \cos\left(2\cos^{-1}\left(1-\frac{s}{4}\right)\right) = 1 - \cos(2\alpha) = 1 - \left[2\cos^2\alpha - 1\right] = 1 - [2(1-s/4)^2 - 1] = s - \frac{s^2}{8}.$$

Problem 17.

Calculate the volume of the region bounded by the parabolic cylinder $x = y^2$, the planes $z = 0$ and $x + z = 1$.

`Answer:` The region is z-simple. The shadow of the region in the (xy) plane is the region R bounded by $x = y^2$ and $x = 1$. For each value of x and y, we have $z_{min} = 0$ and $z_{max} = 1 - x$. We have

$$\text{volume} = \int_R \int_{z_{min}}^{z_{max}} dz dA = \int_{-1}^{1} \int_{y^2}^{1} \int_0^{1-x} dz dx dy =$$

$$= 2 \int_0^1 \int_{y^2}^1 (1-x)\, dx\, dy = 2 \int_0^1 (x - \frac{1}{2}x^2)|_{x=y^2}^{x=1}\, dy = \int_0^1 \left(\frac{1}{2} - y^2 + \frac{1}{2}y^4 \right) dy = \frac{8}{15}.$$

Problem 18.

Find the integral

$$\int\int\int_D |y-1|\, dV$$

where D is the oblique segment of a paraboloid bounded by $z = x^2 + y^2$ and the plane $z = 2y + 3$.

`Answer:` The region is x-simple, so we evaluate

$$\int\int\int |y-1|\, dx\, dz\, dy.$$

The shadow in the yz plane is the region R bounded by the line

$$z = 2y + 3$$

and the parabola

$$z = y^2.$$

These intersect in $(-1, 1)$ and $(3, 9)$. Clearly

$$x_{min} = -\sqrt{z - y^2},\ x_{max} = \sqrt{z - y^2}.$$

We have

$$\int\int_R \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} |y-1|\, dx\, dA = \int_{-1}^{3} \int_{y^2}^{2y+3} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} |y-1|\, dx\, dz\, dy.$$

We proceed to evaluate the integral. After evaluating the inner integral, we are left with

$$2 \int_{-1}^{3} \int_{y^2}^{2y+3} |y-1| \sqrt{z - y^2}\, dz\, dy = 2 \int_{-1}^{3} |y-1| \cdot \frac{2}{3} \left(z - y^2\right)^{\frac{3}{2}} |_{z=y^2}^{z=2y+3}\, dy$$

$$= \frac{4}{3} \int_{-1}^{3} |y-1| (3 + 2y - y^2)^{\frac{3}{2}}\, dy.$$

We can evaluate this integral by substitution. Write

$$u = y - 1$$

obtaining

$$\frac{4}{3} \int_{-2}^{2} |u| (4 - u^2)^{\frac{3}{2}}\, du = \frac{8}{3} \int_0^2 u (4 - u^2)^{\frac{3}{2}}\, du = -\frac{8}{3} (4 - u^2)^{5/2} \cdot \frac{1}{5} |_{u=0}^{u=2} = \frac{256}{15}.$$