MATH 20C - PRACTICE PROBLEMS FOR MIDTERM I

1. Consider the points $P(2,1,0)$, $Q(1,0,1)$ and $R(2,-1,1)$.

(i) Find the area of the triangle PQR.
(ii) Find the equation of the plane through P, Q and R.
(iii) What is the intersection of the plane PQR with the line through $(-1,0,0)$ and parallel to the vector $\vec{i}+\vec{j}+\vec{k}$?
(iv) Find the angle $\angle PQR$.

2. Consider the planes

$$x-2y+2z=-1, \quad x-y+2z=0.$$

(i) Find the angle between the two planes.
(ii) Find the parametric equation for the line of intersection of the two planes.

3. Consider the function

$$f(x,y)=(x-1)^2+(y-2)^2+3.$$

(i) Draw the contour diagram for f. Use at least three different levels.
(ii) Draw the graph of f.
(iii) What does the graph of $g(x,y)=\sqrt{f(x,y)-3}$ look like?

4.

(i) Assume $\vec{x}$ and $\vec{y}$ are vectors of equal length. Show that $\vec{x}+\vec{y}$ and $\vec{x}-\vec{y}$ are perpendicular vectors.
(ii) It is known that $\vec{x}$, $\vec{y}$ and $\vec{z}$ are vectors of length 1 such that the angles between them are

$$\angle(\vec{x},\vec{y})=60, \ \angle(\vec{y},\vec{z})=45, \ \angle(\vec{x},\vec{z})=90.$$

For what value of the parameter a will the vectors $\vec{x}+\vec{y}-a\vec{z}$ and $\vec{x}+\vec{y}$ be orthogonal?

5.

(i) Does the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+4y^4}$$

exist? If yes, what is its value?
(ii) Are there any values of a such that the function

$$f(x,y)=\begin{cases} \frac{x^4y}{x^4+y^4} & \text{if } (x,y)\neq(0,0) \\ a & \text{if } (x,y)=(0,0) \end{cases}$$

is continuous?

6. Consider the curve given by the parametric equations

$$x=e^t\cos t, \quad y=e^t\sin t, \quad 0\le t\le \pi.$$

(i) Find the points (x,y) on the curve where the tangent has slope -1.
(ii) Compute the speed of the curve at any point.
(iii) Calculate the length of the curve.
(iv) Find the arclength parametrization of the curve.
(v) Show that the magnitude of the velocity and acceleration of a particle with the trajectory $(x(t),y(t))$ are connected as

$$||\vec{a}(t)||=\sqrt{2}||\vec{v}(t)||.$$

SOLUTIONS

1.

(i) We calculate the vectors

$$\vec{QP} = (1, 1, -1),\ \vec{QR} = (1, -1, 0).$$

This choice of the two vectors we just computed is motivated by part (iv) below. We know that the area of the parallelogram spanned by $\vec{QP}$ and $\vec{QR}$ is the length of the cross product. We have

$$\vec{QP} \times \vec{QR} = (-1, -1, -2) \implies ||\vec{QP} \times \vec{QR}|| = \sqrt{6}.$$

The triangle PQR has half the area $\frac{1}{2}\sqrt{6}$.

(ii) The normal vector to the plane is the cross product $\vec{QP} \times \vec{QR} = (-1, -1, -2)$. The components of this vector give the coefficients in the equation of the plane. Using any of the three points P, Q or R, we find the plane through P, Q and R has equation

$$x + y + 2z = 3.$$

(iii) The line has the parametric equation

$$(x, y, z) = (-1, 0, 0) + t(1, 1, 1) = (-1 + t, t, t).$$

Substitute into the equation of the plane to find

$$x + y + 2z = 3 \implies (-1 + t) + t + 2t = 3 \implies t = 1.$$

The corresponding intersection point is $(0, 1, 1)$.

(iv) We use the dot product for the vectors $\vec{QP}$ and $\vec{QR}$ to find the angle

$$\cos\theta = \frac{\vec{QP} \cdot \vec{QR}}{||\vec{QP}|| \cdot ||\vec{QR}||} = 0 \implies \theta = 90.$$

2.

(i) The normal vectors are

$$\vec{n}_1 = (1, -2, 2),\ \vec{n}_2 = (1, -1, 2).$$

We use dot product to find the angle between the normal vectors

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{||\vec{n}_1|| \cdot ||\vec{n}_2||}.$$

We calculate

$$\vec{n}_1 \cdot n_2 = 7, ||\vec{n}_1|| = 3, ||\vec{n}_2|| = \sqrt{6}$$

hence

$$\cos\theta = \frac{7}{3\sqrt{6}}.$$

(ii) The line of intersection is perpendicular to both $\vec{n}_1$ and $\vec{n}_2$, hence has direction given by the cross product

$$\vec{n}_1 \times \vec{n}_2 = (-2, 0, 1).$$

We need to find a point on the line of intersection. This can be done by picking a value for z, such as $z = 0$, and solving for x and y:

$$x - 2y = -1,\ x - y = 0 \implies x = y = 1.$$

A point on the line of intersection is $(1, 1, 0)$, and since the line points in the direction $(-2, 0, 1)$ it is given by

$$(x, y, z) = (1, 1, 0) + t(-2, 0, 1) \implies x = 1 - 2t,\ y = 1\ z = t.$$

This is not the only possible answer, picking different values for z gives different final answers.

3.

(i) We have

$$f(x,y) = c \iff (x-1)^2 + (y-2)^2 = c - 3$$

hence the level curve of f for level c is a circle of center $(1,2)$ and radius $\sqrt{c-3}$. We can pick three different values for c such as $c = 4$, $c = 7$ and $c = 12$, and we get three concentric circles, centered at $(1,2)$, of radii 1, 2 and 3 respectively.

(ii) The graph of f is a paraboloid.

(iii) The graph of $g(x,y) = \sqrt{(x-1)^2 + (y-2)^2}$ is a cone with vertex $(1,2,0)$, as we have seen in lecture.

4.

(i) We know $\vec{x}$ and $\vec{y}$ have the same magnitude

$$||\vec{x}|| = ||\vec{y}|| \implies \vec{x} \cdot \vec{x} = \vec{y} \cdot \vec{y}.$$

We find the dot product

$$(\vec{x} - \vec{y}) \cdot (\vec{x} + \vec{y}) = \vec{x} \cdot \vec{x} - \vec{y} \cdot \vec{x} + \vec{x} \cdot \vec{y} - \vec{y} \cdot \vec{y} = \vec{x} \cdot \vec{x} - \vec{y} \cdot \vec{y} = 0.$$

Since the two vectors have zero dot product, they must be perpendicular.

(ii) We know the three vectors have unit length

$$\vec{x} \cdot \vec{x} = \vec{y} \cdot \vec{y} = \vec{z} \cdot \vec{z} = 1$$

and

$$\vec{x} \cdot \vec{y} = ||\vec{x}|| \cdot ||\vec{y}|| \cos 60 = \frac{1}{2}$$
$$\vec{y} \cdot \vec{z} = ||\vec{y}|| \cdot ||\vec{z}|| \cos 45 = \frac{1}{\sqrt{2}}$$
$$\vec{x} \cdot \vec{z} = 0.$$

Since orthogonal vectors have zero dot product, we must have

$$(\vec{x} + \vec{y} - a\vec{z}) \cdot (\vec{x} + \vec{y}) = 0.$$

Cross-multiplying we obtain

$$\vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{x} - a\vec{z} \cdot \vec{x} + \vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} - a\vec{z} \cdot \vec{y} = 1 + \frac{1}{2} - a \cdot 0 + \frac{1}{2} + 1 - a\frac{1}{\sqrt{2}} = 0.$$

We solve $a = 3\sqrt{2}$.

5.

(i) If we let $x, y \to 0$ along the parabola $x = my^2$, the fraction becomes

$$\frac{xy^2}{x^2 + 4y^4} = \frac{my^4}{m^2y^4 + 4y^4} = \frac{m}{m^2 + 4}.$$

This does depend on m, hence the limit does not exist.

(ii) The only possible value for a is

$$a = \lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^4 + y^4}.$$

We claim $a = 0$. Indeed,

$$0 \le \left|\frac{x^4 y}{x^4 + y^4}\right| \le \left|\frac{x^4}{x^4 + y^4}\right| \cdot |y| \le |y| \to 0 \text{ when } (x, y) \to (0, 0).$$

6.

(i) We have

$$dx = e^t(\cos t - \sin t)\, dt,\;\; dy = e^t(\cos t + \sin t)\, dt.$$

The slope equals

$$\frac{dy}{dx} = \frac{e^t(\cos t - \sin t)\, dt}{e^t(\cos t + \sin t)\, dt} = \frac{\cos t - \sin t}{\cos t + \sin t} = -1$$

which implies

$$\cos t - \sin t = -\cos t - \sin t \iff \cos t = 0 \iff t = \frac{\pi}{2}.$$

The corresponding point is

$$x = 0, y = e^{\frac{\pi}{2}}.$$

(ii) We have

$$\vec{r}(t) = (e^t \cos t, e^t \sin t) \implies \vec{r}'(t) = (e^t(\cos t - \sin t), e^t(\cos t + \sin t)).$$

Therefore

$$||r'(t)||^2 = e^{2t}\left[(\cos t - \sin t)^2 + (\cos t + \sin t)^2\right] = e^{2t}(2\cos^2 t + 2\sin^2 t) = 2e^{2t} \implies ||\vec{r}'(t)|| = e^t\sqrt{2}.$$

(iii) We compute

$$\text{length} = \int_0^\pi ||\vec{r}'(t)||\, dt = \int_0^\pi e^t\sqrt{2}\, dt = e^t\sqrt{2}|_{t=0}^{t=\pi} = (e^\pi - 1)\sqrt{2}.$$

(iv) We find the inverse of the arclength function. First the arclength function is

$$s(t) = \int_0^t e^u\sqrt{2}\, du = \sqrt{2}(e^t - 1)$$

hence inverting

$$s(t) = s \implies t = \ln(\frac{s}{\sqrt{2}} + 1).$$

The arclength parametrization is

$$x = e^t \cos t = e^{\ln(\frac{s}{\sqrt{2}}+1)} \cos\ln(\frac{s}{\sqrt{2}} + 1) = (\frac{s}{\sqrt{2}} + 1)\cos\ln(\frac{s}{\sqrt{2}} + 1),$$

$$y = e^t \sin t = e^{\ln(\frac{s}{\sqrt{2}}+1)} \sin\ln(\frac{s}{\sqrt{2}} + 1) = (\frac{s}{\sqrt{2}} + 1)\sin\ln(\frac{s}{\sqrt{2}} + 1),$$

for

$$0 \le s \le (e^\pi - 1)\sqrt{2}.$$

(v) We have $\vec{a}(t) = \vec{r}''(t)$. Since

$$\vec{r}'(t) = (e^t(\cos t - \sin t), e^t(\cos t + \sin t))$$

we find

$$\vec{r}''(t) = (-2e^t \sin t, 2e^t \cos t)$$

which has magnitude

$$||\vec{a}(t)|| = \sqrt{(-2e^t)^2 \sin^r + (2e^t)^2 \cos^t} = 2e^t.$$

We already calculated

$$||\vec{v}(t)|| = e^t\sqrt{2}$$

in part (i), hence

$$||\vec{a}(t)|| = \sqrt{2}||\vec{v}(t)||.$$