MATH 20C - PRACTICE PROBLEMS FOR MIDTERM II

1. Find the critical points of the function

$$f(x, y) = 2x^3 - 3x^2y - 12x^2 - 3y^2$$

and determine their type i.e. local min/local max/saddle point.

2. Determine the global max and min of the function

$$f(x, y) = x^2 - 2x + 2y^2 - 2y + 2xy$$

over the compact region

$$-1 \leq x \leq 1, \ 0 \leq y \leq 2.$$

3. Using Lagrange multipliers, optimize the function

$$f(x, y) = x^2 + (y + 1)^2$$

subject to the constraint

$$2x^2 + (y - 1)^2 \leq 18.$$

4. Consider the function

$$w = e^{x^2 y}$$

where

$$x = u\sqrt{v}, \ y = \frac{1}{uv^2}.$$

Using the chain rule, compute the derivatives

$$\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}.$$

5. Consider the function $f(x, y) = \frac{x^2}{y^4}$.

(i) Carefully draw the level curve passing through $(1, -1)$. On this graph, draw the gradient of the function at $(1, -1)$.

(ii) Compute the directional derivative of f at $(1, -1)$ in the direction $\mathbf{u} = \left(\frac{4}{5}, \frac{3}{5}\right)$. Use this calculation to estimate

$$f((1, -1) + .01\mathbf{u}).$$

(iii) Find the unit direction $\mathbf{v}$ of steepest descent for the function f at $(1, -1)$.

(iv) Find the two unit directions $\mathbf{w}$ for which the directional derivative $D_{\mathbf{w}} f = 0$.

6. Consider the function

$$f(x, y) = x^4 y^3.$$

(i) Write down the equation of the tangent plane at the graph of the function at the point $(1, 1, 1)$.

(ii) Write down an expression for the change, Δz, in $z = f(x, y)$ depending on Δx and Δy, the change in x and y, respectively, near the point $x = y = 1$. Is the function $f(x, y)$ more sensitive to a change in x or to a change in y?

(iii) Using your answer to (ii), find the approximate value of $f(1.01, 1.01)$.

7. Show that the surfaces

$$z = 7x^2 - 12x - 5y^2 \text{ and } xyz^2 = 2$$

intersect orthogonally at the point (2, 1, −1). That is, show that the tangent planes to the two surfaces are perpendicular.

8. Evaluate $\iint_D 3ydA$, where D is the region bounded by

$$xy^2 = 1,\ y = x,\ x = 0,\ y = 3.$$

9. Evaluate

$$\int_0^\pi \int_y^\pi \frac{\sin(x)}{x}\, dxdy.$$

10. Find the volume of the region bounded on top by the plane $z = x + 3y + 1$, on the bottom by the xy-plane, and on the sides by the planes $x = 0$, $x = 3$, $y = 1$, $y = 2$.

11. (*Harder, solve only after looking at problems 1-10*) Two paraboloids

$$z = (x - 2)^2 + (y - 2)^2$$

and

$$z = 20 - x^2 - y^2$$

intersect along a curve C. Find the point of C which is closest to the point $(1, 1, 0)$.

SOLUTIONS

Problem 1.

Find the critical points of the function

$$f(x,y) = 2x^3 - 3x^2y - 12x^2 - 3y^2$$

and determine their type i.e. local min/local max/saddle point. Are there any global min/max?

`Solution:` Partial derivatives

$$f_x = 6x^2 - 6xy - 24x, f_y = -3x^2 - 6y.$$

To find the critical points, we solve

$$f_x = 0 \implies x^2 - xy - 4x = 0 \implies x(x-y-4) = 0 \implies x = 0 \text{ or } x - y - 4 = 0$$

$$f_y = 0 \implies x^2 + 2y = 0.$$

When $x = 0$ we find $y = 0$ from the second equation. In the second case, we solve the system below by substitution

$$x - y - 4 = 0, x^2 + 2y = 0 \implies x^2 + 2x - 8 = 0$$

$$\implies x = 2 \text{ or } x = -4 \implies y = -2 \text{ or } y = -8.$$

The three critical points are

$$(0,0),\ (2,-2),\ (-4,-8).$$

To find the nature of the critical points, we apply the second derivative test. We have

$$A = f_{xx} = 12x - 6y - 24,\ B = f_{xy} = -6x,\ C = f_{yy} = -6.$$

At the point $(0,0)$ we have

$$f_{xx} = -24, f_{xy} = 0, f_{yy} = -6 \implies AC - B^2 = (-24)(-6) - 0 > 0 \implies \boxed{(0,0)\text{is local max}}.$$

Similarly, we find

$$\boxed{(2,-2) \text{ is a saddle point}}$$

since

$$AC - B^2 = (12)(-6) - (-12)^2 =< 0$$

and

$$\boxed{(-4,-8) \text{ is saddle}}$$

since

$$AC - B^2 = (-24)(-6) - (24)^2 < 0.$$

Problem 2.

Determine the global max and min of the function

$$f(x,y) = x^2 - 2x + 2y^2 - 2y + 2xy$$

over the compact region

$$-1 \leq x \leq 1,\ 0 \leq y \leq 2.$$

`Solution:` We look for the critical points in the interior:

$$\nabla f = (2x - 2 + 2y, 4y - 2 + 2x) = (0,0) \implies 2x - 2 + 2y = 4y - 2 + 2x = 0 \implies y = 0, x = 1.$$

However, the point $(1,0)$ is not in the interior so we discard it for now.

We check the boundary. There are four lines to be considered:

- the line $x = -1$:
  $$f(-1,y) = 3 + 2y^2 - 4y.$$
  The critical points of this function of y are found by setting the derivative to zero:
  $$\frac{\partial}{\partial y}(3 + 2y^2 - 4y) = 0 \implies 4y - 4 = 0 \implies y = 1 \text{ with } \boxed{f(-1,1) = 1}.$$
- the line $x = 1$:
  $$f(1,y) = 2y^2 - 1.$$
  Computing the derivative and setting it to 0 we find the critical point $y = 0$. The corresponding point $(1,0)$ is one of the corners, and we will consider it separately below.
- the line $y = 0$:
  $$f(x,0) = x^2 - 2x.$$
  Computing the derivative and setting it to 0 we find $2x - 2 = 0 \implies x = 1$. This gives the corner $(1,0)$ as before.
- the line $y = 2$:
  $$f(x,2) = x^2 + 2x + 4$$
  with critical point $x = -1$ which is again a corner.

Finally, we check the four corners

$$(-1,0), (1,0), (-1,2), (1,2).$$

The values of the function f are

$$\boxed{f(-1,0) = 3}, \boxed{f(1,0) = -1}, \boxed{f(-1,2) = 3}, \boxed{f(1,2) = 7}.$$

From the boxed values we select the lowest and the highest to find the global min and global max. We conclude that

$$\text{global minimum occurs at } (1,0)$$
$$\text{global maximum occurs at } (1,2).$$

Problem 3.

Using Lagrange multipliers, optimize the function

$$f(x,y) = x^2 + (y+1)^2$$

subject to the constraint

$$2x^2 + (y-1)^2 \le 18.$$

`Solution:` We check for the critical points in the interior

$$f_x = 2x, f_y = 2(y+1) \implies (0,-1) \text{ is a critical point }.$$

The second derivative test

$$f_{xx} = 2, f_{yy} = 2, f_{xy} = 0$$

shows this a local minimum with

$$\boxed{f(0,-1) = 0}.$$

We check the boundary

$$g(x,y) = 2x^2 + (y-1)^2 = 18$$

via Lagrange multipliers. We compute

$$\nabla f = (2x, 2(y+1)) = \lambda \nabla g = \lambda(4x, 2(y-1)).$$

Therefore

$$2x = 4x\lambda \implies x = 0 \text{ or } \lambda = \frac{1}{2}$$
$$2(y+1) = 2\lambda(y-1).$$

In the first case $x = 0$ we get

$$g(0,y) = (y-1)^2 = 18 \implies y = 1 + 3\sqrt{2}, 1 - 3\sqrt{2}$$

with values

$$\boxed{f(0, 1+3\sqrt{2}) = (2+3\sqrt{2})^2}, \boxed{f(0, 1-3\sqrt{2}) = (2-3\sqrt{2})^2}.$$

In the second case $\lambda = \frac{1}{2}$ we obtain from the second equation

$$2(y+1) = y - 1 \implies y = -3.$$

Now,

$$g(x,y) = 18 \implies x = \pm 1.$$

At $(\pm 1, -3)$, the function takes the value

$$\boxed{f(\pm 1, -3) = (\pm 1)^2 + (-3+1)^2 = 5.}$$

By comparing all boxed values, it is clear the $(0,-1)$ is the minimum, while $(0, 1+3\sqrt{2})$ is the maximum.

Problem 4.

Consider the function

$$w = e^{x^2 y}$$

where

$$x = u\sqrt{v},\ y = \frac{1}{uv^2}.$$

Using the chain rule, compute the derivatives

$$\frac{\partial w}{\partial u},\ \frac{\partial w}{\partial v}.$$

`Solution:` We have

$$\frac{\partial w}{\partial x} = 2xy\exp(x^2y) = 2u\sqrt{v}\frac{1}{uv^2}\exp\left(u^2v\cdot\frac{1}{uv^2}\right) = \frac{2}{v^{3/2}}\exp\left(\frac{u}{v}\right)$$

$$\frac{\partial w}{\partial y} = x^2\exp(x^2y) = u^2v\exp\left(\frac{u}{v}\right)$$

$$\frac{\partial x}{\partial u} = \sqrt{v},\ \ \frac{\partial x}{\partial v} = \frac{u}{2\sqrt{v}}$$

$$\frac{\partial y}{\partial u} = -\frac{1}{u^2v^2},\ \ \frac{\partial y}{\partial v} = -\frac{2}{uv^3}.$$

Thus

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{2}{v^{3/2}}\exp\left(\frac{u}{v}\right)\cdot\sqrt{v} - u^2v\exp\left(\frac{u}{v}\right)\cdot\frac{1}{u^2v^2} =$$
$$= \frac{2}{v}\exp\left(\frac{u}{v}\right) - \frac{1}{v}\exp\left(\frac{u}{v}\right) = \frac{1}{v}\exp\left(\frac{u}{v}\right).$$

Similarly,

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} = -\frac{u}{v^2}\exp\left(\frac{u}{v}\right).$$

Problem 5.

Consider the function $f(x,y) = \frac{x^2}{y^4}$.

(i) *Carefully draw the level curve passing through* $(1,-1)$. *On this graph, draw the gradient of the function at* $(1,-1)$.

(ii) *Compute the directional derivative of* f *at* $(1,-1)$ *in the direction* $\mathbf{u} = \left(\frac{4}{5},\frac{3}{5}\right)$. *Use this calculation to estimate*

$$f((1,-1) + .01\mathbf{u}).$$

(iii) *Find the unit direction* $\mathbf{v}$ *of steepest descent for the function* f *at* $(1,-1)$.

(iv) *Find the two unit directions* $\mathbf{w}$ *for which the directional derivative* $D_{\mathbf{w}}f = 0$.

`Solution:`

(i) The level is $f(1,1) = 1$. The level curve is

$$f(x,y) = f(1,1) = 1 \implies x^2 = y^4 \implies x = \pm y^2.$$

The level curve is a union of two parabolas through the origin. The gradient

$$\nabla f = \left(\frac{2x}{y^4}, \frac{-4x^2}{y^5} \right) \implies \nabla f(1,-1) = (2,4)$$

is normal to the parabolas.

(ii) We compute

$$f_{\mathbf{u}} = \nabla f \cdot \mathbf{u} = (2,4) \cdot \left(\frac{4}{5}, \frac{3}{5} \right) = 4.$$

For the approximation, we have $f(1,-1) = 1$ and

$$f((1,-1) + .01\mathbf{u}) \approx f(1,-1) + .01 f_{\mathbf{u}} = 1 + .01 \cdot 4 = 1.04.$$

(iii) The direction of steepest decrease is opposite to the gradient. We need to divide by the length to get a unit vector:

$$\mathbf{v} = -\frac{\nabla f}{||\nabla f||} = -\frac{(2,4)}{\sqrt{2^2 + 4^2}} = \left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right).$$

(iv) Write

$$\mathbf{w} = (w_1, w_2).$$

We have

$$f_{\mathbf{w}} = \nabla f \cdot \mathbf{w} = (2,4) \cdot \mathbf{w} = 2w_1 + 4w_2 = 0 \implies w_1 = -2w_2.$$

Since $\mathbf{w}$ has unit length

$$w_1^2 + w_2^2 = 1 \implies (-2w_2)^2 + w_2^2 = 1 \implies w_2 = \pm \frac{1}{\sqrt{5}}.$$

Therefore

$$\mathbf{w} = \pm \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right).$$

Problem 6.

Consider the function

$$f(x,y) = x^4 y^3.$$

(i) *Write down the equation of the tangent plane at the graph of the function at the point* $(1,1,1)$.

(ii) *Write down an expression for the change,* Δz*, in* $z = f(x,y)$ *depending on* Δx *and* Δy*, the change in* x *and* y*, respectively, near the point* $x = y = 1$*. Is the function* $f(x,y)$ *more sensitive to a change in* x *or to a change in* y*?*

(iii) *Using your answer to (ii), find the approximate value of* $f(1.01, 1.02)$.

`Solution:`

(i) We compute

$$f_x = 4x^3y^3 \implies f_x(1,1) = 4$$
$$f_y = 3x^4y^2 \implies f_y(1,1) = 3.$$

The tangent plane is

$$z - 1 = 4(x-1) + 3(y-1) \implies 4x + 3y - z = 6.$$

(ii)

$$\Delta z = 4\Delta x + 3\Delta y.$$

The function is more sensitive to a change in x because the x derivative at $(1,1)$ is higher.

(iii) We have

$$\Delta x = 1.01 - 1 = .01, \Delta y = 1.02 - 1 = .02,$$

hence

$$\Delta z = 4(.01) + 3(.02) = .1.$$

This gives

$$z(1.01, 1.02) = z(1,1) + \Delta z = 1.1 \implies f(1.01, 1.02) \approx 1.1.$$

Problem 7.

Show that the surfaces

$$z = 7x^2 - 12x - 5y^2 \text{ and } xyz^2 = 2$$

intersect orthogonally at the point $(2, 1, -1)$. *That is, show that the tangent planes to the two surfaces are perpendicular.*

`Solution:` The first surface is determined by $z = 7x^2 - 12x - 5y^2$, which can be viewed as the level surface

$$f(x, y, z) = 7x^2 - 12x - 5y^2 - z = 0.$$

The normal vector is given by the gradient at $(2, 1, -1)$:

$$\nabla f = (14x - 12, -10y, -1) = (-16, 10, 1).$$

The second surface is the level set determined by $g(x, y, z) = xyz^2 = 2$. A normal vector at $(2, 1, -1)$ is

$$\nabla g = (yz^2, xz^2, +2xyz) = (1, 2, -4).$$

The dot product of these two normal vectors is 0:

$$\nabla f \cdot \nabla g = (-16, 10, 1) \cdot (1, 2, -4) = 0.$$

Since the normals are perpendicular, the surfaces are orthogonal at $(2, 1, -1)$.

Problem 8.

Evaluate $\iint_D 3ydA$, *where* D *is the region bounded by*

$$xy^2 = 1,\ y = x,\ x = 0,\ y = 3.$$

`Solution:`

$$\iint_D 3y\,dA = \int_0^{\frac{1}{9}}\int_x^3 3y\,dydx + \int_{\frac{1}{9}}^1\int_x^{\frac{1}{\sqrt{x}}} 3y\,dydx = \int_0^{\frac{1}{9}}\left[\frac{3y^2}{2}\right]_x^3 dx + \int_{\frac{1}{9}}^1\left[\frac{3y^2}{2}\right]_x^{\frac{1}{\sqrt{x}}} dx =$$

$$= \int_0^{\frac{1}{9}}\left(\frac{27-3x^2}{2}\right)dx + \int_{\frac{1}{9}}^1\left(\frac{3-3x^3}{2x}\right)dx = \left[\frac{27x-x^3}{2}\right]_0^{\frac{1}{9}} + \left[\frac{3\ln|x|-x^3}{2}\right]_{\frac{1}{9}}^1$$

$$= \boxed{1+3\ln 3}.$$

Problem 9.

Evaluate

$$\int_0^\pi\int_y^\pi \frac{\sin(x)}{x}\,dxdy.$$

`Solution:` Changing the order of integration, we obtain:

$$\int_0^\pi\int_y^\pi \frac{\sin(x)}{x}\,dxdy = \int_o^\pi\int_0^x \frac{\sin(x)}{x}\,dydx = \int_0^\pi\left[y\frac{\sin(x)}{x}\right]_0^x dx = \int_0^\pi \sin(x)dx = \Big[-\cos(x)\Big]_0^\pi$$

$$= \boxed{2}.$$

Problem 10.

Find the volume of the region bounded on top by the plane $z = x + 3y + 1$, *on the bottom by the* xy*-plane, and on the sides by the planes* $x = 0,\ x = 3,\ y = 1,\ y = 2.$

`Solution:`

$$\int_0^3\int_1^2 x+3y+1\,dy\,dx = \int_0^3\left[xy+\frac{3y^2}{2}+y\right]_1^2 dx = \int_0^3 x+\frac{11}{2}\,dx = \left[\frac{x^2}{2}+\frac{11x}{2}\right]_0^3 = \frac{9}{2}+\frac{33}{2}$$

$$= \boxed{21}.$$

Problem 11.

Two paraboloids

$$z = (x-2)^2 + (y-2)^2$$

and

$$z = 20 - x^2 - y^2$$

intersect along a curve C. Find the point of C which is closest to the point $(1, 1, 0)$.

`Solution:` We minimize the function

$$f(x, y, z) = (x-1)^2 + (y-1)^2 + z^2$$

subject to the constraints

$$g_1(x, y, z) = (x-2)^2 + (y-2)^2 - z = 0, \;\; g_2(x, y, z) = 20 - x^2 - y^2 - z = 0.$$

We find

$$\nabla f = 2(x-1, y-1, z)$$

$$\nabla g_1 = (2(x-2), 2(y-2), -1), \;\; \nabla g_2 = (-2x, -2y, -1).$$

Therefore, we must have that

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

which gives

$$2(x-1) = 2(x-2)\lambda - 2x\mu$$

$$2(y-1) = 2(y-2)\lambda - 2y\mu$$

$$2z = -\lambda - \mu.$$

The first equation gives

$$x(\lambda - \mu - 1) = 2\lambda - 1$$

while the second gives

$$y(\lambda - \mu - 1) = 2\lambda - 1.$$

If

$$\lambda - \mu - 1 = 0$$

we must have $2\lambda - 1 = 0$ hence $\lambda = \frac{1}{2}$ and this $\mu = -\frac{1}{2}$. This yields via the third equation $z = 0$ hence $x = y = 2$ because $g_1 = 0$. This set of numbers does not satisfy the second constraint $g_2 = 0$. Thus

$$\lambda - \mu - 1 \neq 0 \implies x = y = \frac{2\lambda - 1}{\lambda - \mu - 1}.$$

The two constraints g_1 and g_2 become

$$z = 2(x-2)^2 = 20 - 2x^2 \implies x^2 + (x-2)^2 = 10 \implies x^2 - 2x = 3 \implies (x-1)^2 = 4 \implies x = -1 \text{ or } 3.$$

When

$$x = y = -1 \implies z = 18$$

which gives $f(-1, -1, 18) = 4 + 4 + 18^2$. When

$$x = y = 3 \implies z = 2$$

which gives $f(3, 3, 2) = 4 + 4 + 4 = 12$. The point we are searching is $(3, 3, 2)$.