Problem 1.

A population $y(t)$ of turtles is growing on an island according to the logistic equation with harvesting

$$\frac{dy}{dt} = y(600 - y) - 50,000, \ y(0) = y_0 > 0.$$ 

(i) Find the critical solutions, indicate their type, draw the phase line and sketch the graphs of some solutions.

(ii) Assume that at time $t = 0$ there are 200 turtles on the island. How many turtles will there be on the island in the long run?

**Answer:**

(i) We find the critical points

$$\frac{dy}{dt} = y(600 - y) - 50,000 = (-y + 100)(y - 500) = 0 \implies y = 100 \text{ and } y = 500.$$ 

The parabola $y(600 - y) - 50,000$ is concave, so the signs are negative for $y < 100$, positive for $100 < y < 500$ and negative for $y > 500$. In particular, the function $y$ is decreasing for $y < 100$, increasing for $100 < y < 500$ and decreasing for $y > 500$. Drawing the phase line and sketching some of the solutions, we see that $y = 100$ repels solutions hence it is an unstable critical point. On the other hand $y = 500$ attracts solutions, hence $y = 500$ is a stable critical point.

(ii) Since $y(0) = 200$ which falls in the interval $(100, 500)$, it follows that the solution converges to the stable critical point

$$\lim_{t \to \infty} y(t) = 500.$$
Problem 2.

Consider the inhomogeneous differential equation

\[ x^2 y'' - xy' + y = x \ln x, \text{ for } x > 0. \]

This problem has three main parts (A), (B), (C), all independent of each other.

(A.) Check that \( y_1 = x \) is a solution to the homogeneous differential equation. We now proceed to find a second solution \( y_2 \) to the homogeneous equation.

(B.1) Show that for any fundamental pair of solutions \((y_1, y_2)\) to the homogeneous equation we must have \( W(y_1, y_2) = Cx \) for some constant \( C \neq 0 \).

(B.2) Set \( y_1 = x \). Consider a second solution \( y_2 \) to the homogeneous equation satisfying the initial values \( y_2(1) = 0, \ y_2'(1) = 1 \).

Show that \( W(y_1, y_2) = x \).

(B.3) Use part (B.2) to show that the solution \( y_2 \) must satisfy

\[ xy_2' - y_2 = x. \]

(B.4) Use (B3) to find a second solution \( y_2 \).

(C) Using the solutions \( y_1 = x \) and \( y_2 = x \ln x \) to the homogeneous equation, find the general solution to the inhomogeneous equation \((*)\) by variation of parameters.

Answer:

(A) We verify that \( y_1 = x \) is a solution by computing \( y_1' = 1, y_1'' = 0 \). Direct computation then shows that the differential equation is verified

\[ x^2 y''_1 - xy'_1 + y_1 = 0. \]

(B1) This follows by Abel’s theorem. We first bring the equation in standard form

\[ y'' - \frac{1}{x} y + \frac{1}{x^2} y = 0. \]

Abel’s theorem states that

\[ W(y_1, y_2) = C \exp\left( \int \frac{1}{x} \, dx \right) = C \exp(\ln x) = Cx \]

as needed.

(B2) We compute

\[ W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} x & y_2 \\ 1 & y'_2 \end{vmatrix} = xy'_2 - y_2. \]

Evaluating at \( x = 1 \) we find

\[ W(y_1, y_2)(1) = 1 \cdot y'_2(1) - y_2(1) = 1 \]
using the initial conditions $y_2(1) = 0, y_2'(1) = 1$. Since we already showed in (B1) that $W(y_1, y_2) = Cx$ it follows

$$W(y_1, y_2)(1) = C \cdot 1 = C$$

from where $C = 1$ by comparing with the preceding equation. Thus $W(y_1, y_2) = x$.

(B3) We showed in part (B2) that

$$W(y_1, y_2) = xy_2' - y_2$$

and

$$W(y_1, y_2) = x$$

from where the conclusion follows.

(B4) To find $y_2$ we use integrating factors. We first write the equation $xy_2' - y_2 = x$ in standard form

$$y_2' - \frac{1}{x}y_2 = 1.$$  

The integrating factor is

$$\mu = \exp\left( - \int \frac{1}{x} \, dx \right) = \exp(-\ln x) = \frac{1}{x}.$$  

Multiplying both sides by the integrating factor we find

$$\left( \frac{1}{x}y_2 \right)' = \frac{1}{x} \Rightarrow \frac{1}{x}y_2 = \ln x + K \Rightarrow y_2 = x \ln x + Kx.$$  

To find the constant $K$ we use the initial value $y_2(1) = 0$ which yields $K = 0$ so that

$$y_2 = x \ln x.$$  

(C) We bring the equation to be solved into standard form

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{\ln x}{x}.$$  

We have computed $W(y_1, y_2) = x$ above. By variation of parameters a particular solution is

$$y_p = u_1y_1 + u_2y_2.$$  

We have

$$u_1 = - \int \frac{\ln x}{x} \cdot \frac{y_2}{W} \, dx = - \int \frac{\ln x}{x} \cdot \frac{x \ln x}{x} \, dx = - \int \frac{(\ln x)^2}{x} \, dx = - \int (\ln x)^2 \cdot (\ln x)' \, dx = - \frac{1}{3} (\ln x)^3.$$  

Similarly,

$$u_2 = \int \frac{\ln x}{x} \cdot \frac{y_1}{W} \, dx = \int \frac{\ln x}{x} \cdot \frac{x \ln x}{x} \, dx = \int \frac{\ln x}{x} \, dx = \int \frac{\ln x \cdot \ln x'}{x} \, dx = \frac{1}{2} (\ln x)^2.$$  

A particular solution is found by substituting into the above expression

$$y_p = - \frac{1}{3} (\ln x)^3 \cdot x + \frac{1}{2} (\ln x)^2 \cdot x \ln x = \frac{1}{6} x (\ln x)^3.$$  

The general solution takes the form

$$y = y_p + y_h = y_p + c_1 y_1 + c_2 y_2 = \frac{1}{6} x (\ln x)^3 + c_1 x + c_2 x \ln x.$$
Problem 3.

Consider the system $\vec{x}' = A\vec{x}$ where

$$A = \begin{bmatrix} -2 & -8 \\ 1 & -8 \end{bmatrix}.$$ 

The eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -6$. (You do not need to check this fact.)

(i) Find a fundamental pair of solutions to the system.

(ii) Draw the trajectories of the general solution. What is the type of the phase portrait you obtained?

(iii) Calculate the matrix exponential $e^{At}$.

(iv) Solve the initial value problem $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(v) Use variation of parameters to find a particular solution to the following inhomogeneous system

$$\vec{x}' = Ax + \begin{bmatrix} 12t \\ 0 \end{bmatrix}.$$ 

Answer:

(i) We find eigenvectors for the two eigenvalues. Letting $A = \begin{bmatrix} -2 & -8 \\ 1 & -8 \end{bmatrix}$ we compute for the first eigenvalue

$$A + 4I = \begin{bmatrix} 2 & -8 \\ 1 & -4 \end{bmatrix} \implies \vec{v}_1 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$ 

For the second eigenvalue, we compute

$$A + 6I = \begin{bmatrix} 4 & -8 \\ 1 & -2 \end{bmatrix} \implies \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$ 

We form the two fundamental solutions

$$\vec{x}_1 = e^{-4t} \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \vec{x}_2 = e^{-6t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$ 

(ii) The general solution is

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 = c_1e^{-4t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2e^{-6t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$ 

When $t \to -\infty$, the solutions are of large magnitude and follow the dominant term $e^{-6t}$ in the direction of the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. When $t \to \infty$, the solutions approach zero, and they follow the dominant term $e^{-4t}$ in the direction $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. The origin is a node sink.

(iii) We have

$$e^{At} = \Phi(t) = \Psi(t) \cdot \Psi(0)^{-1}.$$ 

We find the fundamental matrix

$$\Psi(t) = \begin{bmatrix} 4e^{-4t} & 2e^{-6t} \\ e^{-4t} & e^{-6t} \end{bmatrix}.$$ 

Thus

$$\Psi(0) = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} \implies \Psi(0)^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}.$$
Substituting we find
\[ e^{At} = \begin{bmatrix} 4e^{-4t} & 2e^{-6t} \\ e^{-4t} & e^{-6t} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4e^{-4t} - 2e^{-6t} & -8e^{-4t} + 8e^{-6t} \\ e^{-4t} - e^{-6t} & -2e^{-4t} + 4e^{-6t} \end{bmatrix}. \]

(iv) We have
\[ \vec{x} = e^{At} \cdot \vec{x}_0 = \frac{1}{2} \begin{bmatrix} 4e^{-4t} - 2e^{-6t} & -8e^{-4t} + 8e^{-6t} \\ e^{-4t} - e^{-6t} & -2e^{-4t} + 4e^{-6t} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -4e^{-4t} + 6e^{-6t} \\ -e^{-4t} + 3e^{-6t} \end{bmatrix}. \]

(v) We compute
\[ \vec{x} = \Psi(t) \int \Psi(t)^{-1} \begin{bmatrix} 12t \\ 0 \end{bmatrix} dt. \]

We have
\[ \Psi(t)^{-1} = \frac{1}{2e^{-10t}} \begin{bmatrix} e^{-6t} & -2e^{-6t} \\ -e^{-4t} & 4e^{-4t} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{4t} & -2e^{4t} \\ -e^{6t} & 4e^{6t} \end{bmatrix}. \]
Thus
\[ \vec{x} = \begin{bmatrix} 4e^{-4t} & 2e^{-6t} \\ e^{-4t} & e^{-6t} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} e^{4t} & -2e^{4t} \\ -e^{6t} & 4e^{6t} \end{bmatrix} \cdot \begin{bmatrix} 12t \\ 0 \end{bmatrix} \]
\[ = \begin{bmatrix} 4e^{-4t} & 2e^{-6t} \\ e^{-4t} & e^{-6t} \end{bmatrix} \int \begin{bmatrix} 6te^{4t} \\ -6te^{6t} \end{bmatrix} \]
\[ = \begin{bmatrix} 4e^{-4t} & 2e^{-6t} \\ e^{-4t} & e^{-6t} \end{bmatrix} \cdot \left[ \frac{3}{2} (t - \frac{1}{4}) e^{4t} \right] \]
\[ = \begin{bmatrix} 4t - \frac{7}{6} \\ \frac{1}{2} t - \frac{3}{4} \end{bmatrix}. \]

The integrals were computed via integration by parts. For instance
\[ \int 6te^{6t} dt = \int t(e^{6t})' dt = te^{6t} - \int e^{6t} dt = te^{6t} - \frac{1}{6} e^{6t} = (t - \frac{1}{6}) e^{6t}. \]
The second integral is similar
\[ \int 6te^{4t} dt = \int \frac{3}{2} t(e^{4t})' dt = \frac{3}{2} \left( te^{4t} - \int e^{4t} dt \right) = \frac{3}{2} \left( t - \frac{1}{4} \right) e^{4t}. \]
Problem 4.

Find two independent real valued solutions of the system

\[
\vec{x}' = \begin{bmatrix}
  1 & 1 \\
-5 & 3
\end{bmatrix} \vec{x}.
\]

**Answer:** We write \( A = \begin{bmatrix}
  1 & 1 \\
-5 & 3
\end{bmatrix} \). We compute \( \text{Tr} \ A = 4, \det A = 8 \) so the characteristic polynomial is

\[
\lambda^2 - 4\lambda + 8 = 0 \implies (\lambda - 2)^2 + 4 = 0 \implies \lambda - 2 = \pm 2i \implies \lambda = 2 \pm 2i.
\]

We use only one of the eigenvalues below, say \( \lambda = 2 + 2i \). We find an eigenvector by computing

\[
A - (2 + 2i)I = A = \begin{bmatrix}
-1 - (2 + 2i) & 1 \\
-5 & 3 - (2 + 2i)
\end{bmatrix} = A = \begin{bmatrix}
-1 - 2i & 1 \\
-5 & 1 - 2i
\end{bmatrix} \implies \vec{v} = \begin{bmatrix}
1 \\
1 + 2i
\end{bmatrix}.
\]

Thus a complex valued solution is given by

\[
\vec{x}_1 = e^{(2+2i)t} \begin{bmatrix}
1 \\
1 + 2i
\end{bmatrix} = e^{2t} (\cos 2t + i \sin 2t) \begin{bmatrix}
1 \\
1 + 2i
\end{bmatrix}
\]

\[
= e^{2t} \begin{bmatrix}
\cos 2t + i \sin 2t \\
(1 + 2i)(\cos 2t + i \sin 2t)
\end{bmatrix} = e^{2t} \begin{bmatrix}
\cos 2t + i \sin 2t \\
\cos 2t + i \sin 2t + (2 \cos 2t + \sin 2t)
\end{bmatrix}.
\]

We find the real valued solutions by taking the real and imaginary part of the complex valued solution. We have

\[
u_1 = e^{2t} \begin{bmatrix}
\cos 2t \\
\cos 2t - 2 \sin 2t
\end{bmatrix}, \quad v_1 = e^{2t} \begin{bmatrix}
\sin 2t \\
2 \cos 2t + \sin 2t
\end{bmatrix}.
\]

are the real valued solutions. There are other possible answers here as well.
Problem 5.

Consider the differential equation

\[ y'' - xy' - y = 0 \]

whose solutions are power series in \( x \) centered at \( x_0 = 0 \).

(i) Find the recurrence relation between the coefficients of the power series \( y \).

(ii) Write down the first three non-zero terms in each of the two linearly independent solutions.

(iii) Express the solution involving only even powers of \( x \) in closed form. The final answer should be a familiar exponential. You may need to recall the series expansion

\[ e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots + \frac{y^n}{n!} + \ldots \]

Answer:

(i) We write

\[ y = \sum_{n=0}^{\infty} a_n x^n. \]

We compute

\[ y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \implies xy' = \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n \]

where in the above we used that the term corresponding to \( n = 0 \) is in fact zero \( na_n = 0 \) for \( n = 0 \).

In addition,

\[ y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} a_n x^n \]

where the shift \( n \rightarrow n + 2 \) was done in the last step. Thus

\[ y'' - xy' - y = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n \]

\[ = \sum_{n=0}^{\infty} [a_{n+2}(n+1)(n+2) - na_n - a_n] \cdot x^n \]

\[ = \sum_{n=0}^{\infty} [a_{n+2}(n+1)(n+2) - a_n(n+1)] \cdot x^n. \]

Since \( y'' - xy' - y = 0 \) we conclude

\[ a_{n+2}(n+1)(n+2) - a_n(n+1) = 0 \implies a_{n+2}(n+2) - a_n = 0 \]

for all \( n \).

(ii) We write down the first coefficients of the even solution by using \( n = 0, n = 2 \). We find

\[ 2a_2 - a_0 = 0 \implies a_2 = \frac{a_0}{2} \]

\[ 4a_4 - a_2 = 0 \implies a_4 = \frac{a_2}{2} = \frac{a_0}{2 \cdot 2}. \]
The even solution is
\[ y_{\text{even}} = a_0 + a_2 x^2 + a_4 x^4 + \ldots = a_0 + \frac{a_0}{2} x^2 + \frac{a_0}{2 \cdot 4} x^4 + \ldots = a_0 \left(1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} + \ldots \right). \]

Here we can even set \(a_0 = 1\) if we wish to find an answer without any undetermined constants.

For the odd solution we use \(n = 1\) and \(n = 3\) to find
\[ 3a_3 - a_1 = 0 \implies a_3 = \frac{a_1}{3}, \]
\[ 5a_5 - a_3 = 0 \implies a_5 = \frac{a_3}{5} = \frac{a_1}{3 \cdot 5}. \]

This yields
\[ y_{\text{odd}} = a_1 x + a_3 x^3 + a_5 x^5 + \ldots = a_1 \left(x + \frac{1}{3} x^3 + \frac{1}{3 \cdot 5} x^5 + \ldots \right). \]

Again, we could use \(a_1 = 1\) if we wish to find an answer without any undetermined constants.

(iii) We wish to first the pattern for the even solution. If we continue further with \(n = 4\) we find
\[ 6a_6 - a_4 = 0 \implies a_6 = \frac{a_4}{6} = \frac{a_0}{2 \cdot 4 \cdot 6}, \]
while \(n = 6\) yields
\[ 8a_8 - a_6 = 0 \implies a_8 = \frac{a_6}{8} = \frac{a_0}{2 \cdot 4 \cdot 6 \cdot 8}. \]

The pattern is now clear
\[ a_{2k} = \frac{a_0}{2 \cdot 4 \cdot 6 \ldots \cdot (2k)} = \frac{a_0}{2^k \cdot 1 \cdot 2 \ldots \cdot k} = \frac{a_0}{2^k k!}. \]

Let us set \(a_0 = 1\) since we wish to speak about a specific even solution (which is only unique up to scaling). Then
\[ a_{2k} = \frac{1}{2^k k!}, \]
and
\[ y_{\text{even}} = \sum_{k=0}^{\infty} a_{2k} x^{2k} = \sum_{k=0}^{\infty} \frac{1}{2^k k!} x^{2k} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x^2}{2}\right)^k = e^{x^2}. \]
Problem 6.

Consider the function

\[ h(t) = \begin{cases} 
0 & t < 1 \\
t^2 & 1 \leq t < 2 \\
t^2 + t - 2 & t \geq 2 
\end{cases} \]

(i) Express \( h \) in terms of unit step functions.

(ii) Find the Laplace transform of \( h \). You may leave your answer as a sum of fractions.

Answer:

(i) We have \( h(t) = t^2u_1(t) + (t - 2)u_2(t) \).

(ii) We use that

\[ f(t - c)u_c(t) \mapsto e^{-cs}F(s). \]

In our case, the second term is a direct application (taking \( c = 2 \) and \( f(t) = t \) so that \( F(s) = \frac{1}{s^2} \)), so

\[ (t - 2)u_2(t) \mapsto \frac{e^{-2s}}{s^2}. \]

For the first term, we wish to write

\[ t^2u_1(t) = f(t - 1)u_1(t) \]

for some suitable function \( f \) in order to apply the formula. This means

\[ f(t - 1) = t^2 \implies f(t) = (t + 1)^2 = t^2 + 2t + 1 \implies F(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}. \]

We have

\[ t^2u_1(t) = f(t - 1)u_1(t) \mapsto e^{-s}F(s) = \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2} + \frac{e^{-s}}{s}. \]

Therefore

\[ H(s) = \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2}. \]
Problem 7.

Use Laplace transforms to solve the initial value problem

\[ y'' + 2y' + 5y = e^{-2t}, \quad y(0) = 0, y'(0) = 1. \]

Answer: We have

\[ y'' \mapsto s^2 Y - sy(0) - y'(0) = s^2 Y - 1, \]
\[ y' \mapsto sY - y(0) = sY. \]

The equation to be solved becomes after applying Laplace transform

\[
\begin{align*}
\frac{1}{s^2 + 2s + 5}Y &= 1 + \frac{1}{s+2} \\
\Rightarrow Y &= \frac{1}{s^2 + 2s + 5} + \frac{1}{(s+2)(s^2 + 2s + 5)}.
\end{align*}
\]

We need to compute the inverse Laplace transforms of the above expression. The first term

\[
\frac{1}{s^2 + 2s + 5} = \frac{1}{(s+1)^2 + 4}
\]

has inverse Laplace equal to \( \frac{1}{2} \sin 2t e^{-t} \).

The second term is more difficult. We use partial fractions to write

\[
\frac{1}{(s+2)(s^2 + 2s + 5)} = \frac{A}{s+2} + \frac{Bs + C}{s^2 + 2s + 5}.
\]

Direct computation yields

\[
\begin{align*}
A(s^2 + 2s + 5) + (s + 2)(Bs + C) &= 1 \\
\Leftrightarrow &\quad s^2(A + B) + s(2A + 2B + C) + 5A + 2C = 1 \\
\Leftrightarrow &\quad A + B = 0, 2A + 2B + C = 0, 5A + 2C = 0 \\
\Leftrightarrow &\quad A = \frac{1}{5}, B = -\frac{1}{5}, C = 0.
\end{align*}
\]

Thus

\[
\frac{1}{(s+2)(s^2 + 2s + 5)} = \frac{1}{5} \left( \frac{1}{s+2} - \frac{s}{s^2 + 2s + 5} \right) = \frac{1}{5} \left( \frac{1}{s+2} - \frac{s + 1}{(s+1)^2 + 4} + \frac{1}{(s+1)^2 + 4} \right).
\]

The Laplace inverse equals

\[
\frac{1}{5} \left( e^{-2t} - e^{-t} \cos 2t + \frac{1}{2} e^{-t} \sin 2t \right).
\]

Collecting all terms

\[ y(t) = \frac{1}{2} \sin 2te^{-t} + \frac{1}{5} \left( e^{-2t} - e^{-t} \cos 2t + \frac{1}{2} e^{-t} \sin 2t \right) \]

or simplifying

\[ y(t) = \frac{e^{-2t} + 3}{5} e^{-t} \sin 2t - \frac{1}{5} e^{-t} \cos 2t. \]
**Problem 8.**

Consider the forcing function

\[ h(t) = u_{π}(t) - u_{4π}(t). \]

(i) Solve the following initial value problem using Laplace transform

\[ y'' + y = h(t), \quad y(0) = y'(0) = 0. \]

(ii) Write your solution \( y(t) \) explicitly over each of the three intervals

\[ 0 \leq t < π, \quad π \leq t < 4π, \quad 4π \leq t < ∞. \]

(iii) Draw the graph of the solution you found in (i).

**Answer:**

(i) Using the Laplace of \( u_{c}(t) \mapsto \frac{e^{-cs}}{s} \), we compute

\[ H(s) = \frac{e^{-sπ}}{s} - \frac{e^{-4sπ}}{s}. \]

The Laplace transform of the differential equation becomes

\[ s^2Y + Y = H(s) \implies Y = \frac{H(s)}{s^2 + 1} = \frac{e^{-sπ} - e^{-4sπ}}{s(s^2 + 1)}. \]

We need to find the inverse Laplace transform of this last expression. We first decompose into partial fractions

\[ F(s) = \frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1}. \]

This is the Laplace transform of the function

\[ f(t) = 1 - \cos t. \]

Using that \( e^{cs}F(s) \) has Laplace inverse \( u_{c}(t)f(t - c) \) we have

\[ Y = e^{-sπ}F(s) - e^{-4sπ}F(s) \implies y = u_{π}(t)f(t - π) - u_{4π}(t)f(t - 4π) \]

\[ \implies y = u_{π}(t)(1 - \cos(t - π)) - u_{4π}(t)(1 - \cos(t - 4π)). \]

Using periodicity this can be further simplified to

\[ y = u_{π}(t)(1 + \cos t) - u_{4π}(1 - \cos t). \]

(ii) For \( t < π \) we have \( u_{π}(t) = 0 \) so \( y = 0 \)

- For \( π \leq t < 4π \) we have \( u_{π}(t) = 1 \) but \( u_{4π}(t) = 0 \) so \( y = 1 + \cos t \)

- Finally for \( t > 4π \) we have \( u_{π}(t) = u_{4π}(t) = 1 \) so \( y = 1 + \cos t - (1 - \cos t) = 2\cos t. \)

Thus

\[ y(t) = \begin{cases} 
0 & \text{if } t < π \\
1 + \cos t & \text{if } π \leq t < 4π \\
2\cos t & \text{if } t \geq 4π 
\end{cases} \]