Math 20D - Fall 2017 - Midterm II

(1) Linear second order equations.
   (i) Inhomogeneous equations: General solution $y = y_p + y_h$, where $y_p$ is the particular solution, $y_h$ is the homogeneous solution.
   (ii) Find a particular solution by undetermined coefficients
       $y'' + py' + qy = g(t)$.

       Three cases: $g(t)$ can be exponential, trigonometric, polynomial.
       * For $g(t)$ polynomial, look for $y_p$ as a polynomial with undetermined coefficients. Try to guess its degree first.
       * For trigonometric $g(t)$, look for $y_p = A\cos t + B\sin t$.
       * For exponential case $g(t) = e^{at}$, use $y_p = Cte^{at}$ or $y_p = Ct^2e^{at}$ in such a fashion that you do not replicate homogeneous terms.
       * For a term $g(t) = e^{at} \times$ polynomial or trigonometric function, substitute $y = e^{at}u$, find the differential equation for $u$, then solve for $u$ by undetermined coefficients.
   (iii) Alternatively, you may use variation of parameters
       $y = u_1(t)y_1(t) + u_2(t)y_2(t)$
       where
       $u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)} dt$, $u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)} dt$.

(2) First order systems of equations
   $x' = Ax$.
   (i) Find eigenvalues $\lambda_1$, $\lambda_2$ of $A$:
       $\det(A - \lambda I) = 0$.
       Eigenvectors $v$ are found by solving the system
       $(A - \lambda I)v = 0$.
   (ii) Fundamental pair of solutions: the Wronskian
       $W(x_1, x_2)(t) = \begin{vmatrix} x_1(t) & x_2(t) \end{vmatrix} \neq 0$.
       For a fundamental pair, the general solution is
       $x = c_1x_1 + c_2x_2$.
   (iii) Finding solutions: find eigenvalues $\lambda_1, \lambda_2$ with eigenvectors $v_1$ and $v_2$. General solution
       $x = c_1e^{\lambda_1 t}v_1 + c_2e^{\lambda_2 t}v_2$.
   (iv) Phase portraits. Distinct eigenvalues:
       * saddles (real eigenvalues of opposite sign)
       * nodes (sink or source) (real eigenvalues of same sign)
       * spiral (sink or source) (complex eigenvalues). To find the direction of spirals compute the velocity vector at a point on the trajectory.