

Math 20D - Fall 2011 - Final Exam Topics

(A.) **Modelling** with differential equations:

(i) Mixture problems:  $dQ/dt = \text{rate in} - \text{rate out}$ .

(B.) First order equations

(i) Linear equations:

$$y' + p(t)y = q(t).$$

\* Solve by **integrating factors**. Bring the equation in standard linear form;

\* Integrating factor

$$u(t) = \exp\left(\int p(t) dt\right)$$

\* Multiply by  $u$ , rewrite the equation as

$$(uy)' = uq.$$

(ii) Nonlinear equations

\* **Separable**

$$\frac{dy}{dx} = f(x)g(y).$$

Separate variables, then integrate.

\* **Autonomous equations**

$$\frac{dy}{dx} = f(y).$$

Equilibrium solutions are the roots of  $f$ .

\* **Exact**

$$M(x, y) + N(x, y)y' = 0.$$

Check exactness:

$$M_y = N_x.$$

Find a function  $f$  such that  $f_x = M, f_y = N$ . Set  $f = \text{constant}$ .

(C1.) Second order **homogeneous equations**

$$y'' + p(t)y' + q(t)y = 0.$$

(i) General facts:

\* **superposition**: if  $y_1, y_2$  are solutions,  $c_1y_1 + c_2y_2$  is also a solution.

\* **fundamental pair of solutions**: the Wronskian

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \neq 0.$$

For a fundamental pair, the general solution is

$$y = c_1y_1 + c_2y_2.$$

\* **Abel's theorem**

$$W(y_1, y_2) = C \exp\left(-\int p(t)dt\right).$$

(ii) **Constant coefficient equations**:  $p(t) = p, q(t) = q$ .

\* Characteristic equation  $r^2 + pr + q = 0$ .

\* If the roots  $r_1, r_2$  are real, then the fundamental solutions are

$$y_1 = e^{r_1 t}, y_2 = e^{r_2 t}.$$

\* Complex roots: fundamental solutions are the real and imaginary part of  $e^{r_1 t}$ . If  $r_1 = \alpha + i\beta$ , then

$$y_1 = e^{\alpha t} \cos \beta t, y_2 = e^{\alpha t} \sin \beta t.$$

\* Repeated roots  $r_1 = r_2 = a$ , fundamental solutions

$$y_1 = e^{at}, y_2 = te^{at}.$$

(C2.) **Inhomogeneous second order** equations.

(i) General solution

$$y = y_p + y_h,$$

where  $y_p$  is the particular solution,  $y_h$  is the homogeneous solution.

(ii) Find a particular solution by **undetermined coefficients**

$$y'' + py' + qy = g(t).$$

Three cases:  $g(t)$  can be exponential, trigonometric, polynomial.

\* For  $g(t)$  polynomial, look for  $y_p$  as a polynomial with undetermined coefficients. Try to guess its degree first.

\* For trigonometric  $g(t)$ , look for  $y_p = A \cos t + B \sin t$ .

\* For exponential case  $g(t) = e^{at}$ , use

$$y_p = \frac{e^{at}}{f(a)}$$

with  $f(a) = a^2 + pa + q$ . If  $f(a) = 0$ , look for  $y_p = e^{at}(A + Bt)$  for undetermined  $A, B$ .

\* For a term  $g(t) = e^{at} \times$  polynomial or trigonometric function, substitute  $y = e^{at}u$ , find the differential equation for  $u$ , then solve for  $u$  by undetermined coefficients.

(iii) Alternatively, you may use **variation of parameters**

$$y = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where

$$u_1(t) = - \int \frac{y_2(t)g(t)}{W(y_1, y_2)} dt, u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)} dt.$$

(D.) First order **systems** of equations  $\mathbf{x}' = A\mathbf{x}$ .

(i) Find **eigenvalues**  $\lambda$  of  $A$ :

$$\det(A - \lambda I) = 0$$

or alternatively for  $2 \times 2$  matrices

$$\lambda^2 - \lambda \text{Tr}A + \det A = 0.$$

**Eigenvectors** are found by solving the system

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

(ii) Finding solutions: find eigenvalues  $\lambda_1, \lambda_2$  with eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . General solution

$$\mathbf{x} = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2.$$

(iii) Repeated eigenvalues  $\lambda_1 = \lambda_2 = \lambda$ .

\* **Defective case:** one eigenvector  $\mathbf{v}$ . Solutions

$$\mathbf{x}_1 = e^{\lambda t} \mathbf{v}, \quad \mathbf{x}_2 = e^{\lambda t} (\alpha + t \mathbf{v})$$

where  $\alpha$  is a generalized eigenvector

$$(A - \lambda I)\alpha = \mathbf{v}.$$

(iv) **Phase portraits.**

- \* saddles: real eigenvalues of opposite sign,
- \* nodes (sink or source): real distinct eigenvalues of same sign,
- \* spiral (sink or source) complex eigenvalues. To find the direction of spirals compute the velocity vector at a point on the trajectory.
- \* improper nodes: repeated eigenvalues.

(v) **Fundamental matrix**

$$\Psi(t) = [\mathbf{x}_1 \quad \mathbf{x}_2].$$

The Wronskian is the determinant of the fundamental matrix

$$W(\mathbf{x}_1, \mathbf{x}_2) = \det \Psi(t).$$

The normalized fundamental matrix/**matrix exponential:**

$$\Phi(t) = \Psi(t)\Psi(0)^{-1} = e^{At}.$$

Solution to IVP  $\mathbf{x}' = A\mathbf{x}$  and  $\mathbf{x}(0) = \mathbf{x}_0$  is

$$\mathbf{x} = e^{At} \mathbf{x}_0.$$

(vi) **Undetermined coefficients for systems.**

(vii) **Variation of parameters** for systems

$$\mathbf{x}' = A\mathbf{x} + g(t).$$

A particular solution is given by

$$\mathbf{x}_p = \Psi(t) \int \Psi(t)^{-1} g(t) dt.$$

General solution is

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$$

where  $\mathbf{x}_h$  solves the homogeneous system.

(E.) **Series solutions.**

- (i) Radius of convergence.
- (ii) Finding the recurrence relations between the coefficients.

(F.) **Laplace transform.**

(i) When  $f$  has exponential growth,

$$f \rightsquigarrow F(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

(ii) Laplace transforms of the standard functions:

$$1 \rightsquigarrow \frac{1}{s}, \quad t^n \rightsquigarrow \frac{n!}{s^{n+1}},$$

$$e^{at} \rightsquigarrow \frac{1}{s-a}, \quad e^{at} f(t) \rightsquigarrow F(s-a)$$

$$\sin(at) \rightsquigarrow \frac{a}{s^2 + a^2}, \quad \cos(at) \rightsquigarrow \frac{s}{s^2 + a^2}.$$

(iii) Solving differential equations with Laplace transform:

$$f' \rightsquigarrow sF(s) - f(0),$$

$$f'' \rightsquigarrow s^2F(s) - sf(0) - f'(0).$$

(iv) Discontinuous functions:

$$u_a(t) \rightsquigarrow \frac{e^{-sa}}{s},$$

$$u_a(t)f(t-a) \rightsquigarrow e^{-sa}F(s).$$