Math 220, Problem Set 2. Due Wednesday, October 14.

1. Let $C$ be the unit half circle that joins $1+i$ to $1-i$ clockwise. By direct parametrization, calculate

$$\int_C \sqrt{z-1} \, dz$$

where the principal branch of the square root is used for the integrand.

2. Let $f$ be complex polynomial whose roots are roots $a_1, \ldots, a_k$ with multiplicities $m_1, \ldots, m_k$, that is

$$f(z) = c \prod_{\ell=1}^k (z-a_\ell)^{m_\ell}.$$ 

Let $U \subset \mathbb{C} \setminus \{a_1, \ldots, a_k\}$. Show that for all loops $\gamma$ in $U$, we have

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz = \sum_{\ell=1}^k m_\ell \cdot n(\gamma, a_\ell).$$

This is a version of the more general argument principle to be proven later.

**Remark:** For Problems 3 and 4, you may assume that any complex differentiable function is sufficiently many times differentiable, if it simplifies your argument. We will see later that this is indeed the case.

3. Assume that $f$ is a complex differentiable function. Let $U$ be a connected open set, and let $\gamma$ be a continuous closed curve in $U$.

   (i) If $|f(z) - 1| < 1$ for all $z \in U$ show that

   $$\int_{\gamma} \frac{f''(z)}{f(z)} \, dz = 0.$$ 

   (ii) If $f(z) \neq 0$ for all $z \in U$ and $U$ is simply connected, show that

   $$\int_{\gamma} \frac{f'(z)}{f(z)} \, dz = 0.$$ 

   (iii) If $f(z) \neq 0$ for all $z \in U$ but $U$ is not necessarily simply connected, is it still true that

   $$\int_{\gamma} \frac{f'(z)}{f(z)} \, dz = 0$$

   for all closed curves $\gamma$?

4. Let $n \geq 2$ be a positive integer. Let $U$ be a simply connected open subset of $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be a complex differentiable function with $f(z) \neq 0$. Show that there exists a complex differentiable function $g : U \to \mathbb{C}$ such that

$$g(z)^n = f(z).$$

**Hint:** You may wish to think of the logarithm of $f$. In turn, this has something to do with Problem 3(ii).
5. 

(i) Let \( \gamma \) be a \( C^1 \)-path. Show that the function 
\[ a \mapsto \int_{\gamma} \frac{dz}{z - a} \]
defined for \( a \in \mathbb{C} \setminus \text{Im} \gamma \) is a continuous function.

(ii) Let \( \gamma \) be a \( C^1 \)-loop. Derive from (i) that the winding number 
\[ a \mapsto n(\gamma, a) \]
is constant over each connected component of \( \mathbb{C} \setminus \text{Im} \gamma \).

(iii) Finally, in the setting of (ii), show that \( n(\gamma, a) = 0 \) for \( a \) in an unbounded component of \( \mathbb{C} \setminus \text{Im} \gamma \).

6. Let \( \gamma_1, \gamma_2 \) be two continuous closed paths in \( \mathbb{C} \setminus \{0\} \), and consider the path \( \gamma \) defined by the product 
\[ \gamma(t) = \gamma_1(t) \cdot \gamma_2(t). \]
(Directly from the definition) show that 
\[ n(\gamma, 0) = n(\gamma_1, 0) + n(\gamma_2, 0). \]