Q1. Consider the region $|z| < 2$ and let $f = z^4 + 5z + 3$ and $g = z^4$. Over the boundary circle $|z| = 2$ we have

$$|f - g| = |5z + 3| \leq 5|z| + 3 = 13 < |g| = |z|^4 = 16.$$ 

By Rouché $f$ has as many zeros as $g$ in $|z| < 2$, that is, $f$ has exactly four zeros. When $|z| < 1$, take $f = z^4 + 5z + 3$ and $h = 5z$. In this case, for $|z| = 1$, we have

$$|f - h| = |z^4 + 3| \leq |z|^4 + 3 = 4 < |h| = 5.$$ 

Therefore, $f$ has as many zeroes as $h$ in $|z| \leq 1$, namely one zero. Thus $f$ has 3 zeros in the region $1 < |z| < 2$.

Q2. Let 

$$f(z) = z + e^{-z} - \lambda, \quad g(z) = z - \lambda.$$ 

Consider $\gamma$ the boundary of the half disc of radius $R$ contained in the right half plane $\text{Re} \ z > 0$. We assume that the radius $R > \lambda + 1$. Then, if $z$ is on the half circle, it follows

$$|f - g| = |e^{-z}|$$

$$= e^{-\text{Re}(z)}$$

$$\leq 1 < R - \lambda \leq |z - \lambda| = |g|.$$ 

Furthermore, if $z$ on the diameter of the half circle lying on $y$-axis from $-Ri$ to $Ri$, then it follows

$$|f - g| = |e^{-z}|$$

$$= e^{-\text{Re}(z)}$$

$$= 1 < \lambda \leq \sqrt{\lambda^2 + |\text{Im}(z)|^2} = |g|.$$ 

Hence, by Rouché’s Theorem, $f$ has only one solution inside the half circle contour with a radius $R$. By taking $R \to \infty$, we conclude that $f$ has only one solution on the half plane $\{z : \text{Re}(z) > 0\}$.

Q3. Let $f(z) = z^4 + 3z^2 + z + 1$ and $g(z) = 3z^2 + 1$. For $z$ on the unit circle, it follows

$$|f - g| = |z^4 + z| \leq |z|^4 + |z| \leq 2$$

and

$$|g| = |3z^2 + 1| \geq 3|z|^2 - 1 = 2.$$
Thus
\[ |f - g| \leq |g| \]
on the unit circle.

We claim that equality cannot in fact occur. Assume otherwise. Note that if
\[ |a + b| = |a| + |b| \]
then \( a = bt \) for \( t \) real and nonnegative or \( b = 0 \). (Just let \( t = a/b \), rewrite the above as \( |t + 1| = |t| + 1 \), which implies \( t \in \mathbb{R}_{\geq 0} \)). In our case, we must have equality throughout. In particular, we must have \( |g| = 2 \) so
\[ |3z^2 + 1| + | - 1| = |g| + 1 = 3 = |3z^2|. \]
By our remark, \( z^2 \) is negative real. Since \( |z^2| = 1 \) we must have \( z^2 = -1 \). Thus \( z = \pm i \). However in this case, it can be seen that \( |f - g| = |z^4 + z| = |1 \pm i| = \sqrt{2} \neq 2 \).

Thus
\[ |f - g| < |g| \]
on the unit circle. By Rouche’s Theorem, we conclude that number of roots of \( f \) is the same as number of roots of \( g \) inside the unit disc which is 2.

**Q4.** We claim that \( f = z^n + a_1 z^{n-1} + \ldots + a_n \) has \( n - 1 \) roots in the disc \( |z| < 1 \). Indeed, take \( g = a_1 z^{n-1} \) and compute for \( |z| = 1: \)
\[ |f - g| = |z^n + a_2 z^{n-2} + \ldots + a_n| \leq |z^n| + |a_2| |z|^{n-2} + \ldots + |a_n| = 1 + |a_2| + \ldots + |a_n| \]
\[ < |a_1| = |g|. \]
Thus by Rouche, \( f \) has \( n - 1 \) roots \( z_1, \ldots, z_{n-1} \) with \( |z_i| < 1 \), and one root \( |z_n| > 1 \).

Assume that \( f \) is reducible so that
\[ f = f_1 f_2. \]
Without loss of generality, we may assume \( z_n \) is a root of \( f_2 \). The roots of \( f_1 \) must be among \( z_1, \ldots, z_{n-1} \). As \( f \) is monic, \( f_1 \) is also monic. Writing \( \alpha \in \mathbb{Z} \)
for the free term of \( f_1 \) we must have \( \alpha \) is the product of the roots of \( f_1 \), hence \( |\alpha| < 1 \) by the above discussion regarding the roots of \( f_1 \). This means \( \alpha = 0 \) so \( f_1(0) = 0 \iff f(0) = a_n = 0 \), which is a contradiction.

**Q5.** We compute the partial products \( p_k = \prod_{n=2}^{k} \left( 1 - \frac{1}{n^2} \right) = \prod_{n=2}^{k} \left( \frac{n^2 - 1}{n^2} \right) = \prod_{n=2}^{k} \frac{n-1}{n} \cdot \frac{n+1}{n} = \prod_{n=2}^{k} \frac{n-1}{n} \prod_{n=2}^{k} \frac{n+1}{n} = \frac{1}{k} \cdot \frac{k+1}{2} = \frac{k+1}{2k} \to \frac{1}{2} \) as \( k \to \infty \).

**Q6.**

(i) Let \( f_n(z) = q^n z \). Then \( \sum_{n=1}^{\infty} f_n \) converges absolutely locally uniformly.

Indeed, if \( |z| \leq R \), then \( |f_n| = |q^n z| \leq R|q|^n \) and \( \sum_{n=1}^{\infty} R|q|^n = \frac{R}{1 - |q|} \), so
the uniform convergence over the disc $|z| \leq R$ follows by Weierstrass $M$-test. By the theorem proved in class, the product $Q$ converges absolutely and locally uniformly to an entire function.

(ii) By direct calculation, we have

$$Q(qz) = \prod_{n=1}^{\infty} (1 + q^n \cdot qz) = \prod_{n=1}^{\infty} (1 + q^{n+1}z) = \prod_{n=2}^{\infty} (1 + q^n z).$$

This shows $Q(z) = (1 + qz)Q(qz)$.

(iii) If $Q(z) = \sum_{n=0}^{\infty} a_n z^n$, then $Q(qz) = \sum_{n=0}^{\infty} a_n q^n z^n$. Thus

$$Q(z) = (1+qz)Q(qz) \implies \sum_{n=0}^{\infty} a_n z^n = (1+qz) \sum_{n=0}^{\infty} a_n q^n z^n = a_0 + \sum_{n=1}^{\infty} (a_n q^n + a_{n-1} q^n) z^n.$$

Identifying the coefficients of $z^n$ we find $a_n = a_n q^n + a_{n-1} q^n$ so

$$a_n = a_{n-1} \cdot \frac{q^n}{1-q^n}.$$

Clearly $a_0 = 1$. By induction, the above recursion implies

$$a_n = \frac{q^{n(n+1)/2}}{(1-q)(1-q^2)\ldots(1-q^n)}.$$

This means

$$Q(z) = 1 + \sum_{n=1}^{\infty} a_n z^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{(1-q)(1-q^2)\ldots(1-q^n)} z^n.$$

(iv) When $z = 1$, we obtain

$$\prod_{n=1}^{\infty} (1 + q^n) = 1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{(1-q)(1-q^2)\ldots(1-q^n)}.$$

When $q = t^2$ and $z = t^{-1}$ we obtain

$$\prod_{n=1}^{\infty} (1 + t^{2n-1}) = 1 + \sum_{n=1}^{\infty} \frac{t^{n^2}}{(1-t^2)(1-t^4)\ldots(1-t^{2n})}.$$

**Q7.** Fix $r > 0$. We show uniform convergence of the series of principal logs over the disc $\Delta(0,r)$. To begin, pick $N > 0$ such that $\frac{r}{n} < 1$, for $n \geq N$. Take $z = x + iy \in \Delta(0,r)$. In particular, $|x| \leq r, |y| \leq r$. Therefore, for $n \geq N$ we have

$$\text{Re} \left( 1 + \frac{z}{n} \right) = 1 + \frac{x}{n} \geq 1 - \frac{r}{n} > 0$$

and

$$\text{Re} \left( e^{-\pi z/n} \right) = e^{-\pi} \cos \frac{y}{n} > 0,$$

since $\left| \frac{y}{n} \right| \leq \frac{r}{n} < \frac{\pi}{2}$. Note that

$$\text{Log}(ab) = \text{Log}(a) + \text{Log}(b)$$

when $a, b$ have positive real parts. (Indeed, the identity is clearly true up to multiples of $2\pi i$. It suffices to observe that both sides have argument in the interval
\((-\pi, \pi)\). This is true for the left hand side by definition, while for the right hand side, the arguments of Log \(a\) and Log \((b)\) are in the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\). It follows
\[
\left| \log \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}} \right| = \left| \log \left( 1 + \frac{z}{n} \right) - \log e^{-\frac{z}{n}} \right| = \left| \log \left( 1 + \frac{z}{n} \right) - \frac{z}{n} \right|
\]
\[
= \left| \sum_{m \geq 1} \frac{(-1)^{m+1}}{m} \left( \frac{z}{n} \right)^{m-1} \frac{z}{n} \right|
\]
\[
\leq \sum_{m \geq 2} \left| \frac{(-1)^{m+1}}{m} \left( \frac{z}{n} \right)^{m-1} \frac{z}{n} \right| \leq \sum_{m \geq 2} \left( \frac{r}{n} \right)^{m-1}
\]
\[
= \left( \frac{r}{n} \right)^2 \sum_{m \geq 0} \left( \frac{r}{n} \right)^m \leq \left( \frac{r}{n} \right)^2 \frac{1}{1 - \frac{r}{n}} \leq C_{N,r} \left( \frac{r}{n} \right)^2
\]
where \(C_{N,r} = \frac{r^2}{1 - \frac{r}{n}}\). Since \(\sum_{n \geq N} \frac{1}{n^2}\) converges, by Weierstrass M-test,
\[
\sum_{n=1}^{\infty} \log \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}}
\]
converges absolutely and uniformly on \(\Delta (0, r)\). Hence, as shown in class, the product
\[
\prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}}
\]
converges absolutely and locally uniformly to an entire function \(G(z)\).