HW6 - SOLUTIONS

Q1. If $f = g^n$, then for any zero $a$ of $g$ we have

$$\text{ord}(f, a) = \text{ord}(g^n, a) = n \cdot \text{ord}(g, a)$$

is divisible by $n$. Conversely, if $\text{ord}(f, a)$ is divisible by $n$ for any zero $a$ of $f$, by Weierstraβ, we can construct an entire function $h$ with zeros exactly at the zeros of $f$ and of order $\frac{1}{n} \cdot \text{ord}(f, a)$. Clearly $f$ and $h^n$ have exactly the same zeros with the same multiplicity, hence $f/h^n$ is entire and zero free. In particular,

$$f/h^n = e^F$$

for some entire function $F$. ($F$ is the logarithms of the nowhere zero function $f/h^n$ over the simply connected region $\mathbb{C}$.) Then $f = g^n$ where $g = h \cdot e^{\frac{1}{n} F}$.

Q2. (i) We show that there exist constants $c_1, c_2 > 0$ such that

$$c_1(|t| + 1) \leq |t\omega_1 + \omega_2| \leq c_2(|t| + 1),$$

for all $t$ real. Consider the function $f : \mathbb{R} \to \mathbb{R}, f(t) = \frac{|t\omega_1 + \omega_2|}{|t| + 1}$. We have

$$\lim_{t \to \pm \infty} f(t) = |\omega_1| \neq 0$$

so we can find $\delta$ such that $|f(t)| > |\omega_1|/2$ for $|t| \geq \delta$. Over the interval $[-\delta, \delta]$, $f$ is continuous so it achieves a minimum $m$. Clearly $f(t) \neq 0$ since otherwise $-t\omega_1 = \omega_2$ contradicting the assumption. Therefore $m \neq 0$ and letting $c_1 = \min(m, |\omega_1|/2)$ we conclude $f(t) \geq c_1$ for all $t \in \mathbb{R}$. In a similar fashion we show the existence of $c_2$.

If $n \neq 0$, letting $t = m/n$, we have therefore established that

$$c_1(|m| + |n|) \leq |m\omega_1 + n\omega_2| \leq c_2(|m| + |n|).$$

When $n = 0$, we can arrange that the same inequality hold as well. By the comparison test, $\sum_{\lambda \in \Lambda, \lambda \neq 0} \frac{1}{|\lambda|^\alpha}$ converges if and only if

$$\sum_{m,n \neq 0} \frac{1}{(|m| + |n|)^\alpha}$$

converges. For each $k > 0$, there are $4k$ integer solutions to the equation $|m| + |n| = k$. Thus

$$\sum_{(m,n) \neq (0,0)} \frac{1}{(|m| + |n|)^\alpha} = \sum_{k=1}^{\infty} \frac{1}{k^\alpha} \cdot 4k = \sum_{k=1}^{\infty} \frac{4}{k^{\alpha-1}} < \infty.$$
if and only if $\alpha > 2$. This shows that $\sum_{\lambda \in A, \lambda \neq 0} \frac{1}{|\lambda|^\alpha}$ converges for $\alpha = 3$ but diverges for $\alpha = 2$. Thus $\alpha = 3$ is the smallest integer for which the series converges.

(ii) The fact that the product

$$\prod_{\lambda \in A, \lambda \neq 0} E_2 \left( \frac{z}{\lambda} \right)$$

converges absolutely and locally uniformly was established in class during the proof of the Weierstraß factorization theorem. The convergence of $\sum_{\lambda \in A, \lambda \neq 0} \frac{1}{|\lambda|^\alpha}$ is used to ensure convergence of the product. The statement about the zeros of the $\sigma$-function also follows from the Weierstraß factorization theorem, while the statement about genus follows by (i).

Q3. Clearly,

$$\prod_{n=-\infty}^{\infty} E_1 \left( \frac{z}{n - \alpha} \right)$$

converges absolutely and locally uniformly to an entire function with zeroes only at $n - \alpha$ for $n \in \mathbb{Z}$. This is ensured by the Weierstraß theorem, because the sum

$$\sum_n \frac{1}{|n - \alpha|^2}$$

covers (use the limit comparison test with the series $\sum_n \frac{1}{n^2}$ for instance). The function $\frac{\sin \pi (z + \alpha)}{\sin \pi \alpha}$ also has zeros at $n - \alpha$ for $n \in \mathbb{Z}$. Therefore, by the Weierstraß theorem, we must have

$$\frac{\sin \pi (z + \alpha)}{\sin \pi \alpha} = e^{g(z)} \prod_{n=-\infty}^{\infty} E_1 \left( \frac{z}{n - \alpha} \right).$$

Setting $z = 0$, we see that $e^{g(0)} = 1$ hence $g(0) = 0$. To conclude, it suffices to show

$$g'(z) = \pi \cot \pi \alpha.$$

In the above expression, take the logarithmic derivative. This yields

$$\pi \cot \pi (z + \alpha) = g'(z) + \sum_{n=-\infty}^{\infty} \left( \frac{1}{z - (n - \alpha)} + \frac{1}{n - \alpha} \right).$$

Problem 13 on the midterm practice reads

$$\pi \cot \pi w = \frac{1}{w} + 2w \sum_{n=1}^{\infty} \frac{1}{w^2 - n^2} = \frac{1}{w} + \sum_{n=-\infty, n \neq 0}^{\infty} \left( \frac{1}{w - n} - \frac{1}{w + n} \right)$$

where in the last line we group the terms for $n$ and $-n$ together. With the aid of this identity for $w = z + \alpha$, we obtain

$$g'(z) = \pi \cot \pi (z + \alpha) - \sum_{n=-\infty}^{\infty} \left( \frac{1}{z - (n - \alpha)} + \frac{1}{n - \alpha} \right).$$
\[
\frac{1}{z + \alpha} + \sum_{n=\infty, n \neq 0}^{\infty} \left( \frac{1}{z + \alpha - n} - \frac{1}{n} \right) - \sum_{n=-\infty}^{\infty} \left( \frac{1}{z + \alpha - n} + \frac{1}{n - \alpha} \right)
\]

\[
= \frac{1}{\alpha} + \sum_{n=\infty, n \neq 0}^{\infty} \left( \frac{1}{\alpha - n} - \frac{1}{n} \right) = \pi \cot \pi\alpha,
\]

after applying Problem 13 again.

**Q4.**

(i) Using the triangle inequality, we have

\[
\frac{\alpha + |\alpha z|}{(1 - \bar{\alpha}z)\alpha} \leq \frac{|\alpha| + |\alpha||z|}{|1 - \bar{\alpha}z||\alpha|} = \frac{1 + |z|}{1 - |\bar{\alpha}||z|} \leq \frac{1 + |z|}{1 - |z|} \leq \frac{1 + r}{1 - r}.
\]

By direct calculation

\[
|1 - B_\alpha(z)| = \left| 1 + \frac{z - \alpha}{1 - \bar{\alpha}z}, \frac{\alpha}{\alpha} \right| = \left| \frac{\alpha(1 - \bar{\alpha}z) + (z - \alpha)|\alpha|}{\alpha(1 - \bar{\alpha}z)} \right| = \frac{\alpha - z|\alpha|^2 + z|\alpha| - \alpha|\alpha|}{\alpha(1 - \bar{\alpha}z)} = (1 - |\alpha|) \cdot \frac{\alpha + |\alpha||z|}{(1 - \bar{\alpha})\alpha} \leq \frac{1 + r}{1 - r}(1 - |\alpha|).
\]

(ii) By (i), we have

\[
|1 - B_{\alpha_n}(z)| \leq \frac{1 + r}{1 - r}(1 - |\alpha_n|)
\]

over \(\Delta(0, r)\). By the Weierstrass M-test,

\[
\sum_{n=1}^{\infty} |1 - B_{\alpha_n}(z)|
\]

converges uniformly over \(\Delta(0, r)\), hence locally uniformly in \(\Delta(0, 1)\). By the theorem proved in class

\[
\prod_{n=1}^{\infty} B_{\alpha_n}(z)
\]

converges absolutely and locally uniformly to a function \(B\) holomorphic in \(\Delta(0, 1)\). The same result implies \(B\) has zeros among the zeros of \(B_{\alpha_n}\), namely at \(\alpha_n\).

(iii) Clearly \(B_\alpha\) has a pole at \(z = \frac{1}{\alpha}\), which is outside the unit disc, so \(B_\alpha\) is holomorphic in \(\Delta = \Delta(0, 1)\) and continuous over \(\overline{\Delta}\). We show \(|B_\alpha(z)| = 1\) for \(|z| = 1\). That is, we show \(|z - \alpha| = |1 - \bar{\alpha}z|\) for \(|z| = 1\). When \(|z| = 1\), we have \(\bar{z} = \frac{1}{z}\) so

\[
|z - \alpha| = |\bar{z} - \bar{\alpha}| = \left| \frac{1}{z} - \bar{\alpha} \right| = \left| \frac{1 - z\bar{\alpha}}{z} \right| = |1 - z\bar{\alpha}|
\]

as claimed. By the maximum modulus principle, \(B_\alpha\) achieves its maximum over the boundary \(|z| = 1\), so \(|B_\alpha(z)| < 1\) for \(|z| < 1\). Thus \(B_\alpha\) maps \(\Delta(0, 1)\) to \(\Delta(0, 1)\).
(iv) Assume that 0 is a zero of order $m$ for $f$. Define $g(z) = f(z)/z^m$, so $g$ must have a removable singularity at the origin. Extend $g$ to a holomorphic function over $\Delta(0,1)$. Let $\alpha_1, \ldots, \alpha_n$ be the zeros of $g$, possibly repeated according to multiplicity. We must have only finitely many zeros since if there are infinitely many, they must accumulate in $\Delta(0,1)$. The accumulation point cannot be in $\Delta(0,1)$ since $g$ is holomorphic, so it must lie on the boundary. But by continuity, $g$ must be 0 at this point as well, which is impossible as $|g| = 1$ for $|z| = 1$.

Write $B(z) = \prod_{k=1}^n B_{\alpha_k}(z)$. Then $B$ has zeros at $\alpha_i$ just as $g$. The quotient $h = g/B$ is holomorphic over $\Delta(0,1)$, continuous over $\overline{\Delta}(0,1)$, and it has no zeros. Furthermore, $|B(z)| = 1$ for $|z| = 1$ by part (iii), so $|h(z)| = |g(z)|/|B(z)| = 1$ for $|z| = 1$. By the midterm problem, $h = c$ is constant. Thus $g(z) = cB(z) \implies f(z) = cz^m B(z)$. 