Math 220, Problem Set 2. Due Friday, October 14.

1. Let $C$ be the half unit circle joining $1+i$ to $1-i$ clockwise. By direct parametrization, calculate the integral
$$\int_C \sqrt{z-1} \, dz,$$
where the principal branch of the square root is used for the integrand.

2. Assume that
$$f(z) = c \prod_{\ell=1}^k (z - a_\ell)^{m_\ell}$$
is a polynomial with roots at $a_1, \ldots, a_k$ with multiplicities $m_1, m_2, \ldots, m_k$. Show that for any closed loop $\gamma$ avoiding $a_1, \ldots, a_k$ we have
$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum_{\ell=1}^k m_\ell \cdot n(\gamma, a_\ell).$$

Remark: This is a version of the argument principle to be proved later.

3. Let $f : U \to \mathbb{C}$ be a holomorphic function in a connected open set $U$.
   (i) Assume that $|f(z) - 1| < 1$ for all $z \in U$. Show that
   $$\int_{\gamma} \frac{f'(z)}{f(z)} \, dz = 0$$
   for all closed loops $\gamma$ in $U$.
   (ii) Assume that $U$ is simply connected and $f(z) \neq 0$ for all $z \in U$. Show that
   $$\int_{\gamma} \frac{f'(z)}{f(z)} \, dz = 0$$
   for all closed loops $\gamma$ in $U$.
   (iii) Is it true in general that if $f(z) \neq 0$ for all $z \in U$ then
   $$\int_{\gamma} \frac{f'(z)}{f(z)} \, dz = 0$$
   for all closed loops $\gamma$ in $U$?

4. Assume $f : U \to \mathbb{C}$ is a holomorphic function on a simply connected open set $U$ such that $f(z) \neq 0$ for all $z \in U$. Let $n \geq 2$ be an integer. Show that there is a holomorphic function $g : U \to \mathbb{C}$ such that
$$g(z)^n = f(z).$$

Hint: This has something to do with problem 3(ii).
5.  
(i) Let $\gamma$ be a continuous path. Show that 
\[ a \to \int_{\gamma} \frac{1}{z - a} \] 
is a continuous function in $a \in \mathbb{C} \setminus \text{Im } \gamma$. 
(ii) Show that if $\gamma$ is a continuous loop, then the assignment 
\[ a \to n(\gamma, a) \] 
defines a constant function over each connected component of $\mathbb{C} \setminus \text{Im } \gamma$. 
(iii) In the setting of (ii), show that if $a$ is in an unbounded component of $\mathbb{C} \setminus \text{Im } \gamma$ then 
\[ n(\gamma, a) = 0. \] 

6. Let $\gamma_1, \gamma_2 : [0, 1] \to \mathbb{C} \setminus \{0\}$ be two continuous closed loops and set 
\[ \gamma(t) = \gamma_1(t) \cdot \gamma_2(t). \] 
Show that 
\[ n(\gamma, 0) = n(\gamma_1, 0) + n(\gamma_2, 0). \]