Math 220, Problem Set 3. Due Friday, October 21.

1. Calculate the following integrals:
   
   (i) $\int_{|z|=2} \frac{e^z}{(z-1)(z-3)}dz$
   
   (ii) $\int_{|z|=2} \frac{\sin z}{z^2+1}dz$
   
   (iii) $\int_{|z|=1} \frac{e^z}{(z-2)^2}dz$
   
   (iv) $\int_{|z+2|=a} \frac{dz}{z^2+1}$
   
   (v) $\int_{|z|=1} \frac{dz}{z^5+i(z-4)}$

2. Assume $f$ is an entire function and $p$ is a polynomial such that

   \[ |f(z)| \leq |p(z)| \]

   for $|z|$ sufficiently large. Show that $f$ is a polynomial as well.

3. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.

   (i) Show that if Re $f$ is bounded (from above or below) then $f$ is constant. You may wish to consider the function $g = e^{\pm f}$.

   (ii) Show that if Re $f \leq$ Im $f$ then $f$ is constant. You may wish to consider $g(z) = (1 + i)f(z)$ and use part (i).

4. Consider the power series expansion

   \[ \frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \]

   The expansion holds for $|z| < 2\pi$. The coefficients $B_k$ are called the Bernoulli numbers.

   (i) Find the first non-zero Bernoulli numbers.

   (ii) Prove that $B_{2k+1} = 0$ for all $k \geq 0$.

   (iii) Show that

   \[ 1^p + 2^p + \ldots + N^p = \frac{1}{p+1} \sum_{j=0}^{p+1} (-1)^j B_j \binom{p+1}{j} \cdot N^{p+1-j} \]

   What does this formula give for $p = 1, 2, 3$?

5. Find all entire functions $f : \mathbb{C} \to \mathbb{C}$ such that

   \[ |f(z)|^2 \leq |z| \]
6. Show that a function \( f : \mathbb{C} \to \mathbb{C} \) which is entire and doubly periodic must be constant. A function \( f \) is doubly periodic provided
\[
f(z) = f(z + \omega_1) = f(z + \omega_2)
\]
for complex numbers \( \omega_1, \omega_2 \) such that \( \omega_1/\omega_2 \notin \mathbb{R} \).

7. Let \( f : U \to \mathbb{C} \) be holomorphic over a connected open set \( U \). Assume that for all \( z \in U \), there are integers \( m, n \) (that may depend on \( z \)) such that
\[
f(z)^m = f(z)^n + 1.
\]
Show that \( f \) is constant.

8. Show that if \( f : \mathbb{C} \to \mathbb{C} \) is an entire function such that \( \lim_{z \to \infty} f(z) = \infty \) then \( f \) is a polynomial.

9. Let \( A = \{ z : 0 < |z| < 1 \} \) and \( B = \{ z : 1 < |z| < 2 \} \). Show that there is no bijective holomorphic map \( f : A \to B \).