HW3 - SOLUTIONS

Q1.

(i) We use CIF for \( f(z) = \frac{e^z}{(z-3)^2} \). We have
\[
\int_{|z|=2} \frac{e^z}{(z-1)(z-3)^2} \, dz = \int_{|z|=2} \frac{f(z)}{z-1} \, dz = 2\pi i f(1) = \frac{\pi i e}{2}.
\]

(ii) Let \( f(z) = \sin z \). By CIF we have
\[
\int_{|z|=2} \frac{\sin z}{z+i} \, dz = 2\pi i f(-i) = -2\pi \beta \sin i = \pi (e - e^{-1}).
\]

(iii) The integrand is holomorphic inside \( |z| = 2 \) and hence the integration is 0.

(iv) We compute
\[
\int_{|z+2i|=a} \frac{dz}{z^2+1}.
\]
If \( a < 1 \), then the integral is 0 because the integrand is holomorphic inside \( |z+2i| = a \).

If \( 1 < a < 3 \), then let \( f(z) = \frac{1}{z-i} \). Note that \(-i\) is within the disc \( |z+2i| < a \). By CIF, we have
\[
\int_{|z+2i|=a} \frac{dz}{z+i} = \int_{|z+2i|=a} \frac{f(z)}{z+i} \, dz = 2\pi i f(-i) = -\pi.
\]
If \( a > 3 \), then both \( i \) and \(-i\) are contained within the disc \( |z+2i| < a \). As done in class, we can consider two small circles \( \delta \) and \( \eta \) around \( i \) and \(-i\) and a path \( \gamma \) joining them, and use a homotopy from \( |z+2i| = a \) to \( \delta + \gamma + \eta + (-\delta) \) to conclude, by applying CIF twice, that
\[
\int_{|z+2i|=a} \frac{dz}{z^2+1} = \int_{\delta} \frac{dz}{z^2+1} + \int_{\gamma} \frac{dz}{z^2+1} + \int_{\eta} \frac{1/(z+i)}{z-i} \, dz + \int_{\delta} \frac{1/(z-i)}{z+i} \, dz
\]
\[
= 2\pi i \left( \frac{1}{2i} - \frac{1}{2i} \right) = 0.
\]

(v) Note that
\[
|z^5 - iz - 4| \geq 4 - |z|^5 - |z| \geq 2
\]
on \( |z| = 1 \). Hence, the integrand is holomorphic inside \( |z| = 1 \) and the integration is 0.

Q2. Suppose \( d \) is the degree of \( p(z) \). Let \( a \in \mathbb{C} \). By Cauchy’s formula
\[
f^{(d+1)}(a) = \frac{(d+1)!}{2\pi i} \int_{|z-a|=R} \frac{f(z)}{(z-a)^{d+2}} \, dz.
\]
Since
\[ \lim_{z \to \infty} \frac{p(z)}{(z - a)^d} < \infty \]
it follows
\[ |p(z)| \leq c|z - a|^d \]
for \(|z|\) large, for a constant \(c\). For \(R\) large, we have therefore
\[ |f(z)| \leq |p(z)| \leq cR^d, \]
for \(|z - a| = R\). The integrand is bounded
\[ \left| \frac{f(z)}{(z - a)^d+2} \right| \leq \frac{c}{R^2}. \]
By the basic estimate proved in class
\[ |f^{d+1}(a)| \leq \frac{(d+1)! \cdot c}{2\pi R^2} \cdot 2\pi R = \frac{c(d+1)!}{R}. \]
Hence, \(f^{(d+1)}(a) = 0\) by taking \(R \to \infty\) in the above estimate. Hence, \(f(z)\) is a polynomial with degree at most \(d\).

**Q3.**

(i) Suppose \(\text{Re}(f)\) is bounded above, then there exists \(M \in \mathbb{R}\) such that
\[ \text{Re}(f) \leq M. \]
Hence,
\[ |e^f| = e^{\text{Re}(f)} \leq e^M. \]
By Liouville’s Theorem, it follows that
\[ e^{f(z)} = e^{f(0)}. \]
In other words, for any \(z \in \mathbb{C}\), there exists \(n_z \in \mathbb{N}\) such that
\[ f(z) - f(0) = 2\pi i n_z. \]
By continuity of \(f(z) - f(0)\), we have \(n_z\) is constant, necessarily equal to 0, so \(f(z) = f(0)\). Hence, \(f(z)\) is a constant.

If \(\text{Re}(f)\) is bounded below, then \(-f\) has its real part bounded above.
Hence, \(-f\) is constant which implies \(f\) is constant.

(ii) Since
\[ \text{Re}((1 + i)f) = \text{Re}(f) - \text{Im}(f) \leq 0 \]
and \((1 + i)f\) is entire, \((1 + i)f\) is constant by part (i). Thus \(f\) is constant.
Q4.

(i) We have

\[ z = (e^z - 1) \left( B_0 + B_1 z + B_2 \frac{z^2}{2} + \ldots \right) = \left( z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots \right) \left( B_0 + B_1 z + B_2 \frac{z^2}{2} + \ldots \right) \]

\[ = B_0 z + \left( B_1 + \frac{1}{2} \right) z^2 + \left( B_2 + B_1 + \frac{1}{6} \right) z^3 + \ldots. \]

Thus

\[ B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}. \]

(ii) Consider

\[ f(z) = \frac{z}{e^z - 1} + \frac{z^2}{2} = \sum_{k=0}^{\infty} B'_k \frac{z^k}{k!} \]

where \( B'_k = B_k \) for \( k \neq 1 \) and \( B'_1 = B_1 + \frac{1}{2} \). We claim that \( f \) is an even function, and therefore all the odd powers of \( z \) must come with zero coefficients. This will prove that \( B_{2k+1} = 0 \) for \( k \geq 1 \). To see that \( f \) is even, we compute

\[ f(z) - f(-z) = \frac{z}{e^z - 1} + \frac{z^2}{2} - \frac{-z}{e^{-z} - 1} + \frac{-z^2}{2} = \frac{z}{e^z - 1} - \frac{z}{e^{-z} - 1} + z = 0, \]

where the last equality can be checked by direct computation.

(iii) We consider the expression

\[ e^z + e^{2z} + \ldots + e^{Nz} = e^z \cdot (1 + e^z + \ldots + e^{(N-1)z}) = e^z \cdot \frac{e^{Nz} - 1}{e^z - 1} = \frac{e^{Nz} - 1}{1 - e^{-z}} \]

\[ = \frac{e^{Nz} - 1}{z} \cdot \frac{z}{1 - e^{-z}}. \]

Note that

\[ \frac{e^{Nz} - 1}{z} = \sum_{k=0}^{\infty} N^{k+1} \frac{z^k}{(k+1)!} \]

\[ \frac{z}{1 - e^{-z}} = \sum_{j=0}^{\infty} (-z)^j \frac{B_j}{j!}. \]

We look at the coefficient of \( z^p \) on the left hand side. It equals

\[ \frac{1}{p!} (1^p + \ldots + N^p). \]

The same coefficient on the right hand side equals

\[ \sum_{j+k=p} \frac{N^{k+1}}{(k+1)!} \cdot (-1)^j \frac{B_j}{j!} = \sum_{j} N^{p+1-j} \cdot (-1)^j \frac{B_j}{j!(p+1-j)!}. \]

Matching the two expressions we found gives the result.
Q5. Let \( g(z) = f(z)^2 \). Then

\[ |g(z)| \leq |z|. \]

By problem Q2, we must have \( g \) is a polynomial which can be at most linear (by examining the proof of Q2.) Note \( g(0) = 0 \) by setting \( z = 0 \). Therefore

\[ g(z) = az. \]

Hence \( f(z)^2 = az \). We claim \( a = 0 \) hence \( f = 0 \). Assume otherwise. Note \( f(0) = 0 \). Write \( n \) for the order of 0 as a zero of \( f \). Then 0 is a zero of \( f \) of order \( 2n \). This contradicts the fact that \( az \) has a simple zero at \( z = 0 \).

Q6. Let \( P = \{ t\omega_1 + s\omega_2 : 0 \leq t \leq 1, 0 \leq s \leq 1 \} \) be the parallelogram in \( \mathbb{C}^2 \) spanned by \( \omega_1, \omega_2 \). This is the image of the square

\[ [0,1] \times [0,1] \ni (t,s) \mapsto t\omega_1 + s\omega_2 \in P \]

hence it is compact. The function \( f \) must be bounded on \( P \) by continuity.

Since \( f \) is periodic, \( f \) must be bounded on \( \mathbb{C} \) as well. Indeed, the real vector space spanned by \( \omega_1, \omega_2 \) must be \( \mathbb{R}^2 \) for dimension reasons, hence any \( z \in \mathbb{C} \) can be written in the form \( z = a\omega_1 + b\omega_2 \). Write \( a = n + t, b = m + s \) with \( t, s \in [0,1) \). Then,

\[ f(z) = f(a\omega_1 + b\omega_2) = f(t\omega_1 + s\omega_2) \]

by periodicity. The values of \( f \) were noted to be bounded in the paragraph above, since \( t\omega_1 + s\omega_2 \in P \).

Finally, \( f \) bounded and entire implies \( f \) constant.

Q7. Let \( A_{mn} = \{ w : w^m = w^n + 1 \} \). Thus

\[ \text{Im } f \subset \bigcup_{m,n} A_{m,n}. \]

By the open mapping theorem, if \( f \) is non constant, the image of \( f \) is open, so it contains a disc \( \Delta \). Hence

\[ \Delta \subset \bigcup_{m,n} A_{m,n}. \]

This is a contradiction since \( A_{mn} \) is finite, and hence the union \( \bigcup_{m,n} A_{m,n} \) is countable.

Q8. Assume \( \lim_{z \to \infty} f(z) = \infty \), and set \( g(z) = f(1/z) \), where \( g : \mathbb{C} \setminus \{0\} \to \mathbb{C} \). Note that \( \lim_{z \to 0} g(z) = \infty \). In particular, by results in class, 0 has to be a pole for \( g \). Write the Laurent expansion for \( g \). In \( \mathbb{C} \setminus \{0\} \) we must have

\[ g(z) = \sum_{k=-N}^{\infty} a_k z^k \implies f(z) = g(z^{-1}) = \sum_{k=-N}^{\infty} a_k z^{-k}. \]
Since $f$ is entire, $f$ can be expanded into Taylor series over $\mathbb{C}$, and thus also over $\mathbb{C} \setminus \{0\}$. This means $a_k = 0$ if $k > 0$. Thus

$$f(z) = \sum_{k=-N}^{0} a_k z^{-k}$$

on $\mathbb{C} \setminus \{0\}$ and thus also on $\mathbb{C}$ by the identity theorem. Thus $f$ is a polynomial.

**Q9.** Assume that $f : A \to B$ exists. In particular, $f$ is bounded since $B$ is bounded. By the removable singularity theorem, $f$ extends holomorphically to $g : \{|z| < 1\} \to \mathbb{C}$.

Furthermore, $1 \leq |g(z)| \leq 2$. If equality occurs somewhere, then this contradicts the maximum modulus principle, hence

$$1 < |g(z)| < 2.$$  

This must hold for $z = 0$ as well, hence $g(0) \in B$. Let $g(0) = b$. Since $f : A \to B$ is bijective, we can find $a \in A$ with

$$f(a) = g(0) = b \implies g(a) = g(0) = b.$$  

Let $U, V$ be two open discs around 0 and $a$ which are disjoint. By the open mapping theorem $g(U), g(V)$ are open. Hence $g(U) \cap g(V)$ is open as well. However,

$$g(U) \cap g(V) = \{b\}$$

as one easily checks, which is a contradiction.