HW5 - SOLUTIONS

Q1.

(a) Let

\[ f(z) = \frac{1 - e^{2iz}}{z^2}. \]

The function \( f \) is holomorphic inside the region bounded by

\[ \gamma = C_R \cup S_2 \cup C_r^* \cup S_1, \]

where \( C_r \) and \( C_R \) are the half circles of radii \( r \) and \( R \), and \( S_1, S_2 \) are the segments \([r, R]\) and \([-R, -r]\). The star decorating \( C_r \) indicates the reversed orientation. Hence,

\[ \int_\gamma f(z) \, dz = 0. \]

Moreover,

\[
\int_{S_1} f + \int_{S_2} f = \int_r^R \frac{1 - e^{2ix}}{x^2} \, dx + \int_{-R}^{-r} \frac{1 - e^{2ix}}{x^2} \, dx
= \int_r^R \frac{2 - (e^{2ix} + e^{-2ix})}{x^2} \, dx
= \int_r^R \frac{2 - 2 \cos 2x}{x^2} \, dx
= 4 \int_r^R \frac{\sin^2 x}{x^2} \, dx.
\]

Also,

\[
\left| \int_{C_R} \frac{1 - e^{2iz}}{z^2} \, dz \right| \leq \int_0^\pi \left| \frac{e^{2iRe^{i\theta}}}{R^2} \right| Rd\theta
= \int_0^\pi \frac{1 + e^{-2R \sin \theta}}{R} \, d\theta
\leq \int_0^\pi \frac{1 + \frac{1}{R}}{R} \, d\theta
\leq \frac{2\pi}{R} \to 0 \text{ as } R \to \infty.
\]

Finally,

\[
\int_{C_r} \frac{1 - e^{2iz}}{z^2} \, dz = \int_{C_r} \frac{e^{2iz} - 1 - 2iz}{z^2} \, dz + \int_{C_r} \frac{2i}{z} \, dz
= \int_{C_r} \frac{e^{2iz} - 1 - 2iz}{z^2} \, dz - 2\pi i.
\]
By examining the Taylor expansion, \( \frac{e^{2iz} - 1}{z^2} \) has a removable singularity at \( z = 0 \), and hence \( \frac{e^{2iz} - 1}{z^2} \) can be extended to an entire function. Particularly, \( \frac{e^{2iz} - 1}{z^2} \) is bounded by \( M \) on for \( |z| < r \). Therefore, we can show that
\[
\left| \int_{C_r} \frac{e^{2iz} - 1}{z^2} \, dz \right| \leq Mr\pi \to 0 \text{ as } r \to 0.
\]

By taking \( r \to 0 \) and \( R \to \infty \), it follows
\[
\int_{C_R} \frac{e^{2iz} - 1}{z^2} \, dz = 0.
\]

(b) Let
\[
f(z) = \frac{\log z}{(1 + z^2)^2}
\]
where \( \log(z) := \log |z| + i \arg(z) \) and \(-\frac{\pi}{2} < \arg(z) < \frac{3}{2}\pi \). Using the same contour as in the Q1b, then \( f \) has a pole at \( i \) with multiplicity 2 inside the region enclosed by the contour. It follows
\[
\int_{C_R} f(z) \, dz = 2\pi i \text{Res}(f, i).
\]
Note \( \text{Res}(f, i) = \left. \frac{d}{dz} \frac{\log z}{(1 + z)^2} \right|_{z=i} = \frac{1}{2}(i + \frac{\pi}{2}) \). Also,
\[
\left| \int_{C_R} \frac{\log z}{(1 + z^2)^2} \, dz \right| \leq \int_0^\pi \frac{\log R + i\theta}{(R^2 - 1)^2} R \, d\theta = O \left( \frac{\log R}{R^3} \right) + O \left( \frac{1}{R^3} \right) \to 0 \text{ as } R \to \infty,
\]
and
\[
\left| \int_{C_r} \frac{\log z}{(1 + z^2)^2} \, dz \right| \leq \int_0^\pi \frac{\log r + i\theta}{1} \, r \, d\theta = \pi \left( r \log r + \frac{r}{2} \right) \to 0 \text{ as } r \to 0.
\]

Furthermore,
\[
\int_r^R \frac{\log x}{(1 + x^2)^2} \, dx + \int_{-R}^{-r} \frac{\log x}{(1 + x^2)^2} \, dx = \int_r^R \frac{\log x}{(1 + x^2)^2} \, dx + \int_r^R \frac{\log x + i\pi}{(1 + x^2)^2} \, dx + \int_r^R \frac{1}{(1 + x^2)^2} \, dx.
\]

Note that when \( r \to 0 \) and \( R \to \infty \), we have
\[
\int_r^R \frac{1}{(1 + x^2)^2} \, dx \to i\pi \int_0^\infty \frac{1}{(1 + x^2)^2} \, dx = i\pi \int_0^{\pi/2} \cos^2 \theta \, d\theta = \frac{\pi^2}{4}.
\]

Combing all these together, we have
\[
\int_0^\infty \frac{\log x}{(1 + x^2)^2} = -\frac{\pi}{4}.
\]
(c) Let

\[ f(z) = \frac{z^\alpha}{1 + z^n} = \exp(\alpha \cdot \log(z)) \]

where \( \log \) denotes the branch of the logarithm with argument in \((0, 2\pi)\) (so that we cut along the positive real axis). Let \( \gamma \) be the keyhole contour made up of four curves \( S_R, S_r, L_1 \) and \( L_2 \). The line segments \( L_1, L_2 \) are used at height \( \delta \) and \(-\delta\) respectively, and \( S_r, S_R \) are parts of the circles with radii \( r \) and \( R \). Write \( \xi = e^{\pi i} \). Then \( f \) has all simple poles \( \xi^{2k+1} \) for \( k = 0, \ldots, n-1 \) inside the region enclosed by the contours. Thus,

\[ \int_{\gamma} f(z) \, dz = 2\pi i \sum_{k=0}^{n-1} \text{Res}_\xi \left( f, \xi^{2k+1} \right). \]

We have

\[ \text{Res}_\xi \left( f, \xi^{2k+1} \right) = \frac{g(\xi^{2k+1})}{h'(\xi^{2k+1})} \]

where \( g = z^\alpha \) and \( h = z^n + 1 \). This yields

\[ \text{Res}_\xi \left( f, \xi^{2k+1} \right) = \frac{1}{n} \cdot \frac{(\xi^{2k+1})^\alpha}{(\xi^{2k+1})^{n-1}} = -\frac{1}{n} (\xi^{2k+1})^\alpha \cdot \xi^{2k+1}. \]

We used that \( \xi^n = -1 \) here. We have

\[ (\xi^{2k+1})^\alpha = \exp(\alpha \cdot \log(\xi^{2k+1})) = \exp \left( \alpha \cdot \frac{\pi i (2k+1)}{n} \right). \]

This is valid since \( \frac{2k+1}{n} \pi \in (0, 2\pi) \) as required by the branch we chose. Therefore,

\[ \text{Res}_\xi \left( f, \xi^{2k+1} \right) = -\frac{1}{n} \exp \left( (\alpha + 1) \cdot \frac{\pi i (2k+1)}{n} \right). \]

Thus

\[ \sum_{k=0}^{n-1} \text{Res}_\xi \left( f, \xi^{2k+1} \right) = -\frac{1}{n} \sum_{k=0}^{n-1} \exp \left( (\alpha + 1) \cdot \frac{\pi i (2k+1)}{n} \right) \]

\[ = -\frac{1}{n} \exp \left( (\alpha + 1) \cdot \frac{\pi i}{n} \right) \cdot \frac{\exp((\alpha + 1) \frac{2\pi i n}{n}) - 1}{\exp((\alpha + 1) \frac{2\pi i n}{n}) - 1} \]

\[ = -\frac{1}{n} \cdot \exp(2\pi i) \cdot \frac{\exp ((\alpha + 1) \cdot \frac{\pi}{n}) - 1}{\exp((\alpha + 1) \frac{2\pi i n}{n}) - 1} \]

\[ = -\frac{1}{n} \frac{e^{2\pi i} - 1}{2i \sin \left( \frac{\pi}{n} (\alpha + 1) \right)} \]

Also, since \( \alpha + 1 - n < 0 \) it follows

\[ \left| \int_{S_R} \frac{z^\alpha}{1 + z^n} \, dz \right| \leq \int_0^{2\pi} \frac{R^n}{R^n - 1} \, Rd\theta = O \left( R^{\alpha + 1 - n} \right) \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty. \]
Furthermore, if $|r| < \frac{1}{2}$, then another integration is
\[
\left| \int_{S_r} \frac{z^\alpha}{1 + z^n} \, dz \right| \leq \int_0^{2\pi} \frac{r^\alpha}{1 - r^n} \, rd\theta \leq 4\pi r^{\alpha + 1} \to 0 \text{ as } r \to 0.
\]

Finally, recall that log was chosen so that $0 \leq \text{Arg}(z) < 2\pi$. Then,
\[
\int_{L_1} f(z) = \int_{R} \frac{(x + i\delta)^\alpha}{1 + (x + i\delta)^n} \, dx = \int_{R} \frac{e^{\alpha \log(x + i\delta)}}{1 + (x + i\delta)^n} \, dx
\]
\[
= \int_{R} \frac{e^{\alpha \log|x + i\delta| + \alpha \text{Arg}(x + i\delta)}}{1 + (x + i\delta)^n} \, dx
\]
\[
= \int_{R} \frac{|x + i\delta|^\alpha e^{\alpha \text{Arg}(x + i\delta)} }{1 + (x + i\delta)^n} \, dx
\]
\[
\to \int_0^\infty \frac{x^\alpha}{1 + (x + i\delta)^n} \, dx \text{ as } \delta, r \to 0 \text{ and } R \to \infty.
\]
and
\[
\int_{L_2} f(z) = \int_{R} \frac{(x - i\delta)^\alpha}{1 + (x - i\delta)^n} \, dx = -\int_{R} \frac{e^{\alpha \log(x - i\delta)}}{1 + (x - i\delta)^n} \, dx
\]
\[
= \int_{R} \frac{e^{\alpha \log|x - i\delta| + \alpha \text{Arg}(x - i\delta)}}{1 + (x - i\delta)^n} \, dx
\]
\[
= \int_{R} \frac{|x - i\delta|^\alpha e^{\alpha \text{Arg}(x - i\delta)} }{1 + (x - i\delta)^n} \, dx
\]
\[
\to -e^{2\pi i} \int_0^\infty \frac{x^\alpha}{1 + (x - i\delta)^n} \, dx \text{ as } \delta, r \to 0 \text{ and } R \to \infty.
\]

By combining all these, we have
\[
(1 - e^{2\pi i}) \int_0^\infty \frac{x^\alpha}{1 + x^n} \, dx = 2\pi i \left( -\frac{i}{n} \frac{\xi^{2n\alpha} - 1}{2i \sin \left( \frac{\pi}{n} (\alpha + 1) \right)} \right)
\]
\[
\int_0^\infty \frac{x^\alpha}{1 + x^n} \, dx = \frac{\pi}{n \sin \left( \frac{\pi}{n} (\alpha + 1) \right)}.
\]

Q2. The function
\[
f(z) = \frac{\pi \cot \pi z}{z^2 - a^2} = \frac{\pi \cos \pi z}{\sin(\pi z)(z^2 - a^2)}
\]
has poles at $z = \pm a$ and at $z = n$ for $n \in \mathbb{Z}$. The residues are found by the rules given in class. First,
\[
\text{Res}(f, a) = \frac{\pi \cot \pi z}{(z^2 - a^2)'} \bigg|_{z=a} = \frac{\pi \cot \pi a}{2a}.\]

Similarly,
\[
\text{Res}(f, -a) = \frac{\pi \cot \pi a}{2a}.
\]
The residue at $z = n$, write $g = \frac{\pi \cos \pi z}{z^2 - a^2}$ and $h = \sin(\pi z)$ so that
\[
\text{Res}(f, n) = \text{Res} \left( \frac{g}{h}, n \right) = \frac{g(n)}{h'(n)} = \frac{1}{n^2 - a^2}.
\]
We have
\[
\frac{1}{2\pi i} \int_{\gamma_n} f(z) \, dz = \sum_{m=-n}^{n} \text{Res}(f, m) + \text{Res}(f, a) + \text{Res}(f, -a)
\]
\[
= \frac{1}{a^2} + 2 \sum_{m=1}^{n} \frac{1}{m^2 - a^2} + \frac{\pi \cot \pi a}{a}.
\]
To complete the proof we show that
\[
\lim_{n \to \infty} \int_{\gamma_n} f(z) \, dz = 0.
\]
To this end, we claim that
\[
|\cot \pi z| \leq 3
\]
over \(\gamma_n\). This implies
\[
|f(z)| \leq \frac{3\pi}{|z^2 - a^2|} \leq \frac{3\pi}{n^2 - |a|^2}
\]
using \(|z| \geq n\) and the triangle inequality for \(n\) sufficiently large. Thus by the basic estimate
\[
\left| \int_{\gamma_n} f \, dz \right| \leq \frac{3\pi}{n^2 - |a|^2} \cdot \text{length}(\gamma_n) = \frac{3\pi}{n^2 - |a|^2} \cdot 2(4n + 1) \to 0,
\]
as \(n \to \infty\).

To show the claim, it suffices by the fact that cotangent is odd, to consider only two sides of the rectangle, for instance the sides:
\[
y = n, \quad |x| \leq n + \frac{1}{2} \quad \text{and} \quad x = n + \frac{1}{2}, \quad |y| \leq n.
\]
We compute
\[
|\cot \pi z| = \left| \frac{e^{\pi iz} + e^{-\pi iz}}{e^{\pi iz} - e^{-\pi iz}} \right| = \left| \frac{e^{2\pi iz} + 1}{e^{-2\pi iz} - 1} \right| = \left| 1 + \frac{2}{e^{-2\pi iz} - 1} \right| \leq 1 + \frac{2}{|e^{-2\pi iz} - 1|}.
\]
We will show \(|e^{-2\pi iz} - 1| > 1\) over the two sides, completing the argument. Indeed, over the side \(y = n\), we have
\[
|e^{-2\pi iz} - 1| = |e^{-2\pi i(n + 1/2)} - 1| \geq e^{2\pi n} - 1 > 1.
\]
Over the side \(x = n + \frac{1}{2}\), we have
\[
|e^{-2\pi iz} - 1| = |e^{-2\pi i(n+1/2) + 2\pi y} - 1| = |e^{2\pi y} - 1| = e^{2\pi y} + 1 > 1.
\]

Q3. The function
\[
f(z) = \frac{\pi \cot \pi z}{(z + a)^2} = \frac{\pi \cos \pi z}{\sin(\pi z)(z + a)^2}
\]
has poles at \(z = -a\) and at \(z = n\) for \(n \in \mathbb{Z}\). The residues are found by the rules given in class. First,
\[
\text{Res}(f, -a) = \frac{d}{dz} \pi \cot \pi z|_{z=-a} = -\frac{\pi^2}{\sin^2 \pi a}.
\]
The residue at $z = n$, write $g = \pi \cos \pi z$ and $h = \sin(\pi z)(z + a)^2$ so that

$$\text{Res}(f, n) = \text{Res}\left(\frac{g}{h}, n\right) = \frac{g(n)}{h'(n)} = \frac{1}{(n + a)^2}.$$ 

We have

$$\frac{1}{2\pi i} \int_{\gamma_n} \frac{\pi \cot \pi z}{(z + a)^2} \, dz = \sum_{n \in \mathbb{Z}} \text{Res}(f, n) + \text{Res}(f, -a) = \frac{1}{(n + a)^2} - \frac{\pi^2}{\sin^2 \pi a}.$$ 

To complete the proof we show that

$$\lim_{n \to \infty} \int_{\gamma_n} f(z) \, dz = 0.$$ 

To this end, we recall by the previous problem that

$$|\cot \pi z| \leq 3$$

over $\gamma_n$. This implies

$$|f(z)| \leq \frac{3\pi}{|z + a|^2} \leq \frac{3\pi}{(n - |a|)^2}$$

using $|z| \geq n$ and the triangle inequality $|z - a| \geq |z| - |a| \geq n - |a| > 0$ for $n$ sufficiently large. Thus by the basic estimate

$$\left| \int_{\gamma_n} f \, dz \right| \leq \frac{3\pi}{(n - |a|)^2} \cdot \text{length}(\gamma_n) = \frac{3\pi}{(n - |a|)^2} \cdot 2(4n + 1) \to 0,$$

as $n \to \infty$.

**Q4.**

(i) We have

$$\lim_{|z| \to \infty} R(z)z^2 = \lim_{|z| \to \infty} \frac{P(z)z^2}{Q(z)} = \alpha$$

where $\alpha$ denotes the quotient of leading terms in $P$ and $Q$ if $\deg P + 2 = \deg Q$ and $\alpha = 0$ otherwise. Thus, for $|z| > \eta$ we have

$$|R(z)z^2| < \alpha + 1 \implies |R(z)| \leq \frac{\alpha + 1}{|z|^2} = \frac{M_2}{|z|^2}.$$ 

To show the claim about the cotangent, it suffices by the fact that cotangent is odd, to consider only two sides of the rectangle, for instance the sides:

$$y = n + \frac{1}{2}, \quad |x| \leq n + \frac{1}{2} \quad \text{and} \quad x = n + \frac{1}{2}, \quad |y| \leq n + \frac{1}{2}.$$ 

We compute

$$|\cot \pi z| = \frac{|e^{\pi iz} + e^{-\pi iz}|}{|e^{\pi iz} - e^{-\pi iz}|} = \frac{e^{2\pi iz} + 1}{e^{2\pi iz} - 1} = 1 + \frac{2}{e^{2\pi iz} - 1} \leq 1 + \frac{2}{|e^{-2\pi iz} - 1|}.$$ 

We will show $|e^{-2\pi iz} - 1| > 1$ over the two sides, completing the argument. Indeed, over the side $y = n + \frac{1}{2}$, we have

$$|e^{-2\pi iz} - 1| = |e^{-2\pi iz}e^{2\pi(n + 1/2)} - 1| \geq |e^{2\pi(n + 1/2)} - 1| > 1.$$
Over the side \( x = n + \frac{1}{2} \), we have

\[
|e^{-2\pi iz} - 1| = |e^{-2\pi i(n+1/2)+2\pi y} - 1| = |e^{2\pi y} - 1| = e^{2\pi y} + 1 > 1.
\]

(ii) Using (i) we have

\[
|R(z)\pi \cot \pi z| \leq \frac{M_1 M_2}{|z|^2} \leq \frac{M_1 M_2}{(n + 1/2)^2}.
\]

Thus

\[
\left| \int_{\gamma_n} R(z)\pi \cot \pi z \right| \leq \frac{M_1 M_2}{(n + 1/2)^2} \cdot \text{length} (\gamma_n) = \frac{M_1 M_2}{(n + 1/2)^2} \cdot 4(2n + 1) \to 0 \quad \text{as} \quad n \to \infty.
\]

(iii) We have \( |R(m)| \leq \frac{M_2}{m^2} \) for \( m \) large, so \( \sum_{m=-\infty}^{\infty} R(m) \) converges by the comparison test, since \( \sum_{m=-\infty}^{\infty} \frac{1}{m^2} \) converges.

(iv) Clearly \( \cot \pi z = \frac{\cos \pi z}{\sin \pi z} \) has poles whenever \( \sin \pi z = 0 \) so for \( z = m, \ m \in \mathbb{Z} \).

By the rules of computing residues, we have

\[
\text{Res}_{z=m} (\pi \cot \pi z) = \text{Res}_{z=m} \left( \frac{\pi \cos \pi z}{\sin \pi z} \right) = \left. \frac{\pi \cos \pi z}{(\sin \pi z)'} \right|_{z=m} = \frac{\pi \cos \pi z}{\pi \cos \pi z} |_{z=m} = 1.
\]

The function \( R(z)\pi \cot \pi z \) has poles at \( z = m \) and at \( z = a_j \). Clearly, since \( R \) is holomorphic near \( m \), we have

\[
\text{Res}_{z=m} (\pi \cot \pi z R(z)) = \text{Res}_{z=m} \left( \frac{1}{z - m} + \ldots \right) (R(m) + (z-m)R'(m) + \ldots) = R(m).
\]

Similarly, since \( \cot \pi z \) is holomorphic near \( a_j \), and \( R \) has a simple pole at \( a_j \) with residue \( \frac{P(a_j)}{Q'(a_j)} \), we have

\[
\text{Res}_{z=a_j} (\pi \cot \pi z R(z)) = \pi \cot \pi a_j \cdot \frac{P(a_j)}{Q'(a_j)}.
\]

Putting everything together via the residue theorem,

\[
0 = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma_n} R(z)\pi \cot \pi z \, dz = \sum_{m=-\infty}^{\infty} R(m) + \sum_{j=1}^{q} \pi \cot \pi a_j \cdot \frac{P(a_j)}{Q'(a_j)}.
\]

This is what we set out to prove.

**Q5.**

(i) If \( f \) has a removable singularity at \( \infty \) then \( g \) has a removable singularity at \( 0 \) hence it is bounded

\[
|g| \leq M
\]

in some disc \( \Delta(0, R) \). Thus for \( |f| \leq M \) for \( |z| \geq \frac{1}{R} \). Since \( f \) is bounded over \( \Delta(0, 1/R) \) by continuity, this means \( f \) is bounded over \( \mathbb{C} \), hence constant by Louiville’s theorem.

(ii) Since \( f \) is entire on \( \mathbb{C} \), it admits an expansion

\[
f(z) = \sum_{n \geq 0} a_n z^n.
\]
Hence
\[ g(z) = \sum_{n \geq 0} a_n \frac{1}{z^n} \]
and since this function must be meromorphic at 0, it follows that all but finitely many \( a_n \) must vanish. Thus \( f \) is a polynomial, as claimed.

(iii) We note that \( f(A) = g(\{z : 0 < |z| < 1/R\}) \). Since \( g \) has an essential singularity at 0, the set \( g(\{z : 0 < |z| < 1/R\}) \) is dense in \( \mathbb{C} \) by Casorati-Weierstrass.

**Q6.** Suppose \( h \) is a meromorphic function on \( \mathbb{C} \cup \{\infty\} \). We first show that \( h \) can only have finitely many zeroes and poles. In fact, it suffices to argue for the poles since by working with \( \frac{1}{h} \) instead we can derive the same statement for the zeros. Assume that \( h \) has infinitely many poles \( a_j \in \mathbb{C} \cup \{\infty\} \).

- if \( a_j \) is a bounded sequence, then \( a_j \) will have a convergent subsequence but this contradicts the fact that the poles of a meromorphic are discrete (by definition);
- if \( a_j \) is unbounded, then \( a_j \) will have a subsequence converging to \( \infty \), again contradicting that the poles of a meromorphic function are discrete in \( \mathbb{C} \cup \{\infty\} \).

We now show that \( h \) is a rational function. Let \( (q_1, \ldots, q_n) \) be the poles (counting multiplicity) of \( h \) on \( \mathbb{C} \). Then let
\[ \phi(z) = h(z) \prod_{j}(z - q_j). \]
This function has no poles on \( \mathbb{C} \), hence it is holomorphic on \( \mathbb{C} \) and has possibly a pole at \( \infty \). By Q5, such a function \( \phi \) is necessarily a polynomial, completing the proof.