HW5 - SOLUTIONS

Q1. Consider the region $|z| < 2$ and let $f = z^4 + 5z + 3$ and $g = z^4$. Over the boundary circle $|z| = 2$ we have

$$|f - g| = |5z + 3| \leq 5|z| + 3 = 13 < |g| = |z|^4 = 16.$$ 

By Rouché $f$ has as many zeros as $g$ in $|z| < 2$, that is, $f$ has exactly four zeros. When $|z| < 1$, take $f = z^4 + 5z + 3$ and $h = 5z$. In this case, for $|z| = 1$, we have

$$|f - h| = |z^4 + 3| \leq |z|^4 + 3 = 4 < |h| = 5.$$ 

Therefore, $f$ has as many zeroes as $h$ in $|z| \leq 1$, namely one zero. Thus $f$ has 3 zeros in the region $1 < |z| < 2$.

Q2. Let

$$f(z) = z + e^{-z} - \lambda, \quad g(z) = z - \lambda.$$ 

Consider $\gamma$ the boundary of the half disc of radius $R$ contained in the right half plane $\text{Re } z > 0$. We assume that the radius $R > \lambda + 1$. Then, if $z$ is on the half circle, it follows

$$|f - g| = |e^{-z}| = e^{-\text{Re}(z)}$$

$$\leq 1 < R - \lambda \leq |z - \lambda| = |g|.$$ 

Furthermore, if $z$ on the diameter of the half circle lying on $y$-axis from $-Ri$ to $Ri$, then it follows

$$|f - g| = |e^{-z}| = e^{-\text{Re}(z)}$$

$$= 1 < \lambda \leq \sqrt{\lambda^2 + |\text{Im}(z)|^2} = |g|.$$ 

Hence, by Rouché’s Theorem, $f$ has only one solution inside the the half circle contour with a radius $R$. By taking $R \to \infty$, we conclude that $f$ has only one solution on the half plane $\{z : \text{Re } z > 0\}$.

Q3. Let $f(z) = z^4 + 3z^2 + z + 1$ and $g(z) = 3z^2 + 1$. For $z$ on the unit circle, it follows

$$|f - g| = |z^4 + z| \leq |z|^4 + |z| \leq 2$$

and

$$|g| = |3z^2 + 1| \geq 3|z|^2 - 1 = 2.$$
Thus

$$|f - g| \leq |g|$$

on the unit circle.

We claim that equality cannot in fact occur. Assume otherwise. Note that if

$$|a + b| = |a| + |b|$$

then $a = bt$ for $t$ real and nonnegative or $b = 0$. (Just let $t = a/b$, rewrite the above as $|t + 1| = |t| + 1$, which implies $t \in \mathbb{R}_{\geq 0}$). In our case, we must have equality throughout. In particular, we must have $|g| = 2$ so

$$|3z^2 + 1| + |1 - 1| = |g| + 1 = 3 = |3z^2|.$$

By our remark, $z^2$ is negative real. Since $|z^2| = 1$ we must have $z^2 = -1$. Thus $z = \pm i$. However in this case, it can be seen that $|f - g| = |z^4 + z| = |1 \pm i| = \sqrt{2} \neq 2$.

Thus

$$|f - g| < |g|$$

on the unit circle. By Rouché’s Theorem, we conclude that number of roots of $f$ is the same as number of roots of $g$ inside the unit disc which is 2.

**Q4.** We claim that $f = z^n + a_1 z^{n-1} + \ldots + a_n$ has $n - 1$ roots in the disc $|z| < 1$. Indeed, take $g = a_1 z^{n-1}$ and compute for $|z| = 1$:

$$|f - g| = |z^n + a_2 z^{n-2} + \ldots + a_n| \leq |z|^n + |a_2||z|^{n-2} + \ldots + |a_n| = 1 + |a_2| + \ldots + |a_n| < |a_1| = |g|.$$

Thus by Rouché, $f$ has $n - 1$ roots $z_1, \ldots, z_{n-1}$ with $|z_i| < 1$, and one root $|z_n| > 1$.

Assume that $f$ is reducible so that

$$f = f_1 f_2.$$

Without loss of generality, we may assume $z_n$ is a root of $f_2$. The roots of $f_1$ must be among $z_1, \ldots, z_{n-1}$. As $f$ is monic, $f_1$ is also monic. Writing $\alpha \in \mathbb{Z}$ for the free term of $f_1$ we must have $\alpha$ is the product of the roots of $f_1$, hence $|\alpha| < 1$ by the above discussion regarding the roots of $f_1$. This means $\alpha = 0$ so $f_1(0) = 0 \implies f(0) = a_n = 0$, which is a contradiction.

**Q5.**

(i) Let $f_n(z) = q^n z$. Then $\sum_{n=1}^{\infty} f_n$ converges absolutely locally uniformly. Indeed, if $|z| \leq R$, then $|f_n| = |q^n z| \leq R |q|^n$ and $\sum_{n=1}^{\infty} R |q|^n = \frac{R}{1-|q|}$, so the uniform convergence over the disc $|z| \leq R$ follows by Weierstrass M-test. By the theorem proved in class, the product $Q$ converges absolutely and locally uniformly to an entire function.
(ii) By direct calculation, we have

\[ Q(qz) = \prod_{n=1}^{\infty} (1 + q^n \cdot qz) = \prod_{n=1}^{\infty} (1 + q^{n+1}z) = \prod_{n=2}^{\infty} (1 + q^nz). \]

This shows \( Q(z) = (1 + qz)Q(qz). \)

(iii) If \( Q(z) = \sum_{n=0}^{\infty} a_n z^n \), then \( Q(qz) = \sum_{n=0}^{\infty} a_n q^n z^n \). Thus

\[ Q(z) = (1 + qz)Q(qz) \implies \sum_{n=0}^{\infty} a_n z^n = (1 + qz) \sum_{n=0}^{\infty} a_n q^n z^n = a_0 + \sum_{n=1}^{\infty} (a_n q^n + a_{n-1} q^n) z^n. \]

Identifying the coefficients of \( z^n \) we find \( a_n = a_n q^n + a_{n-1} q^n \) so

\[ a_n = a_{n-1} \cdot \frac{q^n}{1 - q^n}. \]

Clearly \( a_0 = 1 \). By induction, the above recursion implies

\[ a_n = \frac{q^n(n+1)/2}{(1 - q)(1 - q^2) \ldots (1 - q^n)}. \]

This means

\[ Q(z) = 1 + \sum_{n=1}^{\infty} a_n z^n = 1 + \sum_{n=1}^{\infty} \frac{q^n(n+1)/2}{(1 - q)(1 - q^2) \ldots (1 - q^n)} z^n. \]

(iv) When \( z = 1 \), we obtain

\[ \prod_{n=1}^{\infty} (1 + q^n) = 1 + \sum_{n=1}^{\infty} \frac{q^n(n+1)/2}{(1 - q)(1 - q^2) \ldots (1 - q^n)}. \]

When \( q = t^2 \) and \( z = t^{-1} \) we obtain

\[ \prod_{n=1}^{\infty} (1 + t^{2n-1}) = 1 + \sum_{n=1}^{\infty} \frac{t^{n^2}}{(1 - t^2)(1 - t^4) \ldots (1 - t^{2n})}. \]

Q6. Fix \( r > 0 \). We show uniform convergence of the series of principal logs over the disc \( \Delta(0, r) \). To begin, pick \( N > 0 \) such that \( \frac{r}{n} < 1 \), for \( n \geq N \). Take \( z = x + iy \in \Delta(0, r) \). In particular, \( |x| \leq r, |y| \leq r \). Therefore, for \( n \geq N \) we have

\[ \text{Re} \left( 1 + \frac{z}{n} \right) = 1 + \frac{x}{n} \geq 1 - \frac{r}{n} > 0 \]

and

\[ \text{Re} \left( e^{-\frac{y}{n}} \right) = e^{-\frac{y}{n}} \cos \frac{y}{n} > 0, \]

since \( |\frac{y}{n}| \leq \frac{r}{n} < \frac{\pi}{2} \). Note that

\[ \text{Log}(ab) = \text{Log}(a) + \text{Log}(b) \]

when \( a, b \) have positive real parts. (Indeed, the identity is clearly true up to multiples of \( 2\pi i \). It suffices to observe that both sides have argument in the interval
This is true for the left hand side by definition, while for the right hand side, the arguments of \( \log a \) and \( \log (b) \) are in the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\). It follows

\[
\left| \log \left[ \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}} \right] \right| = \left| \log \left( 1 + \frac{z}{n} \right) - \log e^{-\frac{z}{n}} \right| = \left| \log \left( 1 + \frac{z}{n} \right) - \frac{z}{n} \right|
\]

\[
= \sum_{m \geq 1} \left| \frac{(z/n)^m}{m} \right| \leq \sum_{m \geq 2} \left| \frac{(z/n)^m}{m} \right| \leq \sum_{m \geq 2} \left( \frac{r^n}{m} \right)^m
\]

\[
= \left( \frac{r^n}{n} \right)^2 \sum_{m \geq 0} \left( \frac{r^n}{m} \right)^m \leq \left( \frac{r^n}{n} \right)^2 \frac{1}{1 - \frac{r^n}{n}} \leq C_{N,r} \left( \frac{1}{n} \right)^2
\]

where \( C_{N,r} = \frac{r^n}{n} \). Since \( \sum_{n \geq N} \frac{1}{n^2} \) converges, by Weierstraß M-test,

\[
\sum_{n=1}^{\infty} \log \left[ \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}} \right]
\]

converges absolutely and uniformly on \( \Delta (0, r) \). Hence, as shown in class, the product

\[
\prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}}
\]

converges absolutely and locally uniformly to an entire function \( G(z) \).

Q7.

(i) Using the triangle inequality, we have

\[
\left| \frac{\alpha + |\alpha|z}{1 - \alpha\bar{z}} \right| \leq \frac{|\alpha| + |\alpha||z|}{|1 - \alpha\bar{z}|} = \frac{1 + |z|}{|1 - \alpha\bar{z}|} \leq 1 + |z| \leq \frac{1 + |z|}{1 - |\alpha||z|} \leq 1 + \frac{r}{1 - |\alpha||z|}.
\]

By direct calculation

\[
|1 - B_\alpha(z)| = 1 + \frac{z - \alpha}{1 - \alpha z} \cdot \left| \frac{|\alpha|}{\alpha} \right| = \left| \frac{\alpha(1 - \alpha z) + (z - \alpha)\alpha|\alpha|}{\alpha(1 - \alpha z)} \right| = \left| \frac{\alpha - \alpha \alpha z + z|\alpha| - \alpha|\alpha|}{\alpha(1 - \alpha z)} \right| = \left| \frac{\alpha - z|\alpha|^2 + z|\alpha| - \alpha|\alpha|}{\alpha(1 - \alpha z)} \right| = \left| \frac{1 - |\alpha|^2 + z|\alpha| - \alpha|\alpha|}{\alpha(1 - \alpha z)} \right| \leq \frac{1 + r}{1 - \frac{r}{1 - |\alpha||z|}}.
\]

(ii) By (i), we have

\[
|1 - B_{\alpha_n}(z)| \leq \frac{1 + r}{1 - \frac{r}{1 - |\alpha_n|}}
\]

over \( \Delta(0, r) \). By the Weierstraß M-test,

\[
\sum_{n=1}^{\infty} |1 - B_{\alpha_n}(z)|
\]

converges uniformly over \( \Delta(0, r) \), hence locally uniformly in \( \Delta(0, 1) \). By the theorem proved in class

\[
\prod_{n=1}^{\infty} B_{\alpha_n}(z)
\]
converges absolutely and locally uniformly to a function $B$ holomorphic in $\Delta(0,1)$. The same result implies $B$ has zeros among the zeros of $B_{\alpha_n}$, namely at $\alpha_n$.

(iii) Clearly $B_{\alpha}$ has a pole at $z = \frac{1}{\bar{\alpha}}$ which is outside the unit disc, so $B_{\alpha}$ is holomorphic in $\Delta = \Delta(0,1)$ and continuous over $\overline{\Delta}$. We show $|B_{\alpha}(z)| = 1$ for $|z| = 1$. That is, we show $|z - \alpha| = |1 - \bar{\alpha}z|$ for $|z| = 1$. When $|z| = 1$, we have $\bar{z} = \frac{1}{z}$ so

$$|z - \alpha| = |\bar{z} - \bar{\alpha}| = \left|\frac{1}{z} - \frac{1}{\bar{\alpha}}\right| = \left|\frac{1 - z\bar{\alpha}}{z}\right| = |1 - z\bar{\alpha}|$$

as claimed. By the maximum modulus principle, $B_{\alpha}$ achieves its maximum over the boundary $|z| = 1$, so $|B_{\alpha}(z)| < 1$ for $|z| < 1$. Thus $B_{\alpha}$ maps $\Delta(0,1)$ to $\Delta(0,1)$.

(iv) Assume that 0 is a zero of order $m$ for $f$. Define $g(z) = f(z)/z^m$, so $g$ must have a removable singularity at the origin. Extend $g$ to a holomorphic function over $\Delta(0,1)$. Let $\alpha_1, \ldots, \alpha_n$ be the zeros of $g$, possibly repeated according to multiplicity. We must have only finitely many zeros since if there are infinitely many, they must accumulate in $\overline{\Delta}(0,1)$. The accumulation point cannot be in $\Delta(0,1)$ since $g$ is holomorphic, so it must lie on the boundary. But by continuity, $g$ must be 0 at this point as well, which is impossible as $|g| = 1$ for $|z| = 1$.

Write $B(z) = \prod_{k=1}^n B_{\alpha_k}(z)$. Then $B$ has zeros at $\alpha_i$ just as $g$. The quotient $h = g/B$ is holomorphic over $\Delta(0,1)$, continuous over $\overline{\Delta}(0,1)$, and it has no zeros. Furthermore, $|B(z)| = 1$ for $|z| = 1$ by part (iii), so $|h(z)| = |g(z)|/|B(z)| = 1$ for $|z| = 1$. By the midterm problem, $h = c$ is constant. Thus $g(z) = cB(z) \implies f(z) = cz^m B(z)$. 

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