Math 220, Practice problems for the final exam.

To review, we list below the Main Topics covered in this class (this is not a comprehensive list):

4. Taylor and Laurent series.
5. Zeros of holomorphic functions, open mapping theorem, maximum modulus principle, Liouville’s theorem.
8. The argument principle. Applications to elliptic functions. Rouché’s theorem.

1. Let \( U \) be open and connected, and let \( f, g \) be holomorphic functions such that \( f(z)g(z) = 0 \). Show that either \( f \) or \( g \) is identically zero on \( U \).

2. (i) Let \( x \in \mathbb{C} \). Show that the Laurent expansion
   \[
   \exp \left( \frac{1}{2} x \left( z - \frac{1}{z} \right) \right) = J_0(x) + \sum_{n=1}^{\infty} J_n(x) \left( z^n + \frac{(-1)^n}{z^n} \right)
   \]
   holds for \( 0 < |z| < \infty \) for some coefficients \( J_n(x) \) that depend on \( x \).

   Remark: These coefficients \( J_n \) are called the Bessel functions of the first kind, and appear for instance in the study of the wave equation.

   (ii) Using the expansion of the exponential, show that \( J_n \) are entire and
   \[
   J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left( \frac{x}{2} \right)^{n+2k}.
   \]

   (iii) Show that \( y = J_n(x) \) is a solution to the Bessel differential equation
   \[
   x^2 y'' + xy' + (x^2 - n^2)y = 0.
   \]

3. Assume \( f \) and \( g \) are meromorphic functions on \( \mathbb{C} \) such that
   \[
   |f(z) - g(z)| < |g(z)|
   \]
   for all \( z \in \mathbb{C} \) which are not poles for \( f \) or \( g \). Show that \( f = cg \) for some constant \( c \). One possible approach is via removable singularity theorem.
4. Assume that $f$ and $g$ are entire and $f \circ g = 0$. Show that either $f = 0$ or $g$ is constant. You may wish to recall a problem from Homework 4.

5. Show that there exists a holomorphic function $f$ on the unit disc $\Delta(0,1)$ that cannot be extended to a holomorphic function in the neighborhood of any point on the boundary $\partial \Delta$.

Hint: Let $a_n$ be a sequence in $\Delta(0,1)$ such that each $z \in \partial \Delta(0,1)$ is an accumulation point of $a_n$. Why does such a sequence exist? Use Weierstraβ.

6. Compute the following integrals:
   (i) \[
   \int_{0}^{\infty} \frac{x^a}{1 + x^2} \cdot \frac{dx}{x}, \quad 0 < a < 2. \]
   (ii) \[
   \int_{-\infty}^{\infty} \cos x \frac{dx}{x^\alpha + a^2} \]
   (iii) \[
   \int_{0}^{\infty} \frac{(\log x)^2}{1 + x^2} dx \]
   (iv) \[
   \int_{0}^{\infty} \frac{x^2}{x^4 + x^2 + 1} dx. \]

7. Consider $\omega_1, \omega_2$ two complex numbers with $\omega_1/\omega_2 \notin \mathbb{R}$. Let
   \[
   \Lambda =: \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}. \]
   The Weierstraβ $\sigma$ function is defined as
   \[
   \sigma(z) = z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right) \cdot \exp \left(\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}\right) = z \prod_{\lambda \in \Lambda \setminus \{0\}} E_2\left(\frac{z}{\lambda}\right). \]
   (i) Show that $\sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{|\lambda|^3}$ converges. To this end, show that
   \[
   |m\omega_1 + n\omega_2| \geq c(|m| + |n|) \]
   for some $c > 0$ and all real numbers $n, m$. Show that the equation $|n| + |m| = k$ has $4k$ integral solutions. Conclude that
   \[
   \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{|\lambda|^3} \leq 4c^{-3} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty. \]
   (ii) Show that $\sigma$ is an entire function with zeroes only at the points of $\Lambda$.

8. Exhibit an entire function which vanishes only at the positive integers $n$ with order $n$, and is non-zero everywhere else.
9. Assume that $f$ and $g$ are entire functions. Show that there exist entire functions $h$, $F$ and $G$ such that
\[ f(z) = h(z)F(z), \quad g(z) = h(z)G(z) \]
with $F, G$ having no common zeroes. This is an application of the Weierstrass problem.

10. Possibly using the product expansion of the sine function, prove that
\[ e^{az} - e^{bz} = ze^{(a+b)z/2} \prod_{n=1}^{\infty} \left( 1 + \frac{(a-b)^2 z^2}{4n^2 \pi^2} \right). \]

11. Prove that the polynomial $az^5 + z + 1$ has at least one root inside the disc $|z| \leq 2$.

Hint: there are two cases $a \leq \frac{1}{32}$ and $a > \frac{1}{32}$. One can be treated using Rouché. The other case follows by looking at the product of roots.

12. Let $f(z)$ be a holomorphic function in the disk $|z| < 2$. Show that $\sum_{n=1}^{\infty} \frac{f^{(n)}(z)}{n!}$ converges locally uniformly over $|z| < 1$. You may wish to recall Cauchy’s estimates.

13. Assume that $f : \Delta(0,2) \to \mathbb{C}$ is a holomorphic function such that $|f(z)| \leq M$ for $|z| = 1$. Using Cauchy’s formula, show that for all $w_1, w_2 \in \overline{\Delta}(0, \frac{1}{2})$ we have
\[ |f(w_1) - f(w_2)| \leq 4M|w_1 - w_2|. \]

14.

(i) Let $A = \{ z : |z| \leq R \}$, and let $f$ be a holomorphic function in a neighborhood of $A$. Show that for all $\epsilon > 0$, there exists a polynomial $p$ such that
\[ \sup_{z \in A} |p(z) - f(z)| < \epsilon. \]

(ii) Assume that $A = \{ z : r \leq |z| \leq R \}$ for $R > r > 0$. Show that there exists $\epsilon > 0$ such that for all polynomials $p$ we have
\[ \sup_{z \in A} \left| p(z) - \frac{e^z}{z} \right| > \epsilon. \]

That is, show that $\frac{e^z}{z}$ cannot be approximated by polynomials uniformly on $A$. This is an application of integration.

15.

(i) Show that $C(z) = \frac{z+1}{z+1}$ takes the upper half plane bijectively onto the unit disc; in particular, if $\text{Im} \ z > 0$ then $|C(z)| < 1$.

(ii) Conclude from (i) that there are no entire functions with $f : \mathbb{C} \to \mathbb{C}$ such that $\text{Im} \ f(z) > 0$ for all $z \in \mathbb{C}$.

Remark: This was part of an older problem set, but solved in a different way then.

16. Let $f : \Delta(0,1) \to \Delta(0,1)$ such that $f(0) = 0$. 

(i) Show that $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

Hint: You may wish to consider the function $g(z) = f(z)/z$ for $z \neq 0$, $g(0) = f'(0)$ and use the maximum modulus principle over any disc $\Delta(0, r)$. Make $r \to 1$.

(ii) Show that if $f$ has a fixed point $\alpha \neq 0$ inside $\Delta(0, 1)$ then $f$ is the identity.

17. Show that if $f$ is holomorphic on $U$ and Re $f$ admits a local maximum, then $f$ is constant.

18. In class, we stated the following result. Assume that
\[
\sum_{n=1}^{\infty} |f_n - 1|
\]
converges locally uniformly, and let
\[
h = \prod_{n=1}^{\infty} f_n.
\]
Show that
\[
\frac{h'}{h} = \sum_{n=1}^{\infty} \frac{f_n'}{f_n}
\]
holds away from the zeros of $h$.

(i) Show that for finite products $h = f_1 f_2 \ldots f_k$ then
\[
\frac{h'}{h} = \sum_{n=1}^{k} \frac{f_n'}{f_n}
\]

(ii) By considering the partial $P_n$ products defining $h$, (starting perhaps at the $N^{\text{th}}$ index), and the Weierstraß convergence theorem, derive the result stated above.

19. (Harder.) Fix $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. Using logarithmic derivatives and an identity proved in Homework 5, show that
\[
\frac{\sin \pi(z + \alpha)}{\sin \pi \alpha} = e^{\pi z \cot \pi \alpha} \prod_{n=-\infty}^{\infty} E_1 \left( \frac{z}{n-\alpha} \right).
\]