HW 2 - SOLUTIONS

Q1. We have the following generalized hypergeometric series

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}. $$

where $(a)_n = a(a+1)\cdots(a+n-1)$. The following are special cases :

- ${}_0F_0(;;z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$
- Note that

  $$ (3/2)_n = \frac{3 \cdot 5 \cdots (2n+1)}{2^n} = \frac{(2n+1)!}{4^n (n)!} $$

  Substituting the above expression, we see that

  $$ \begin{aligned} {}_0F_1(;\frac{3}{2};\frac{-z^2}{4}) &= \sum_{n=0}^{\infty} \frac{4^n n!}{(2n+1)!} \frac{(-1)^n z^{2n}}{4^n n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!} = \frac{\sin z}{z} \end{aligned} $$

- The next part is very similar, we have

  $$ (1/2)_n = \frac{1 \cdot 3 \cdots (2n-1)}{2^n} = \frac{(2n)!}{4^n (n)!}, $$

  which we substitute to obtain

  $$ \begin{aligned} {}_0F_1(;\frac{3}{2};\frac{-z^2}{4}) &= \sum_{n=0}^{\infty} \frac{4^n n!}{(2n)!} \frac{(-1)^n z^{2n}}{4^n n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \cos z \end{aligned} $$

- Similarly substituting the values of $(1)_n = n!$ and $(2)_n = (n+1)!$ in

  $$ \begin{aligned} z \cdot {}_2F_1(1,1;2;-z) &= z \cdot \sum_{n=0}^{\infty} \frac{(n!)^2}{(n+1)!} \frac{(-z)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{(n+1)} = \ln(1+z) \end{aligned} $$

Q2. The dilogarithm is defined as

$$ \mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} $$

(i) Show that $\mathrm{Li}_2(z)$ has radius of convergence $R = 1$. Show that Li_2 can be expressed as a generalized hypergeometric series

$$ \mathrm{Li}_2(z) = z \cdot {}_3F_2(1,1,1;2,2;z). $$

Proof. The radius of convergence $R = \lim_{n\to\infty} n^{\frac{-2}{n}} = 1$. Using the fact $(1)_n = n!$ and $(2)_n = (n+1)!$ we get the required equality :

$$z \cdot {}_3F_2(1,1,1;2,2;-z) = z \cdot \sum_{n=0}^{\infty} \frac{n!n!n!}{(n+1)!(n+1)!} \frac{z^n}{n!}$$
$$= \sum_{n=1}^{\infty} \frac{z^n}{n^2} = \mathrm{Li}_2(z).$$

□

(ii) Show that Li_2 is injective in $\Delta(0, \frac{2}{3})$.

Proof. Let $z, w \in \Delta(0, \frac{2}{3})$, then

$$\mathrm{Li}_2(z) - \mathrm{Li}_2(w) = \sum_{n=1}^{\infty} \frac{z^n - w^n}{n^2}$$
$$= (z-w)\left(1 + \sum_{n=2}^{\infty} \frac{(z^{n-1} + \cdots + w^{n-1})}{n^2}\right)$$

To show Li_2 is injective, it is enough to show that

$$1 + \sum_{n=2}^{\infty} \frac{(z^{n-1} + z^{n-2}w + \cdots + w^{n-1})}{n^2} \neq 0$$

for any pair of numbers $z, w \in \Delta(0, \frac{2}{3})$. We show this by proving that the summation in the above expression can not attain absolute value 1.

Using triangle inequality and the fact $z, w \in \Delta(0, \frac{2}{3})$, we obtain the inequality

$$|z^{n-1} + z^{n-2}w + \cdots + w^{n-1}| < n\left(\frac{2}{3}\right)^{n-1},$$

and further use this and uniform convergence to get

$$\left|\sum_{n=2}^{\infty} \frac{(z^{n-1} + z^{n-2}w + \cdots + w^{n-1})}{n^2}\right| < \sum_{n=2}^{\infty} \frac{1}{n}\left(\frac{2}{3}\right)^{n-1}$$
$$< \sum_{n=2}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^{n-1}$$
$$= 1.$$

□

Q3. Show that the function $u : \mathbb{C}\backslash\{0\} \to \mathbb{R}$ given by

$$u(z) = \log|z|$$

is harmonic, but it is not the real part of a holomorphic function in $\mathbb{C}\backslash\{0\}$.

Proof. Let $\mathrm{Log}(z)$ be the standard branch of logarithm defined on $\mathbb{C}^- = \mathbb{C}\backslash\mathbb{R}_{\leq 0}$, and let $L_2 : \mathbb{C}^+ \to \mathbb{C}$ be another branch of logarithm, branched as positive real line. Note that $e^{L_2(z)} = z$, hence $|e^{L_2(z)}| = e^{\mathrm{Re}(L_2(z))} = |z|$, therefore $\mathrm{Re}(L_2(z)) = \log|z|$. The same hold for $\mathrm{Log}(z)$.

For any $z \neq 0$, either $z \in \mathbb{C}^-$ or $\mathbb{C}^+$. On any of these open sets, $u(z)$ can be realized as the real part of a holomorphic function, Log or L_2 respectively, hence u is harmonic.

Suppose f is holomorphic on $\mathbb{C}\backslash\{0\}$ such that $f = u + iv$, where $u(z) = \log|z|$. Therefore,

$$g(z) := (f - \mathrm{Log})(z) \in \{w : \mathrm{Re}(w) = 0\}$$

on the open set $\mathbb{C}^-$. Using Cauchy-Riemann condition one can show that $g(z)$ is a constant κ. Hence on the open set $\mathbb{C}^-$, we have the identity

$$f(z) - \kappa = \mathrm{Log}(z).$$

Thus existence of f is equivalent to extending Log to $\mathbb{C}\backslash\{0\}$, which we know is not possible. □

Q4. Let $f : \mathbb{C} \to \mathbb{C}$ be the inversion $f(z) = \frac{1}{z}$.

(i) Let C be the circle of radius r centered at the origin. Show that $f(C)$ is a circle of radius $1/r$ centered at the origin.

Proof. Let C' be the circle of radius $1/r$ centered at the origin. For any $z \in C$, $f(z) \in C'$ since $|f(z)| = \frac{1}{|z|} = \frac{1}{r}$. Similarly, for any $w \in C'$, $f(w) \in C$ for any z in C or C'. Moreover we know that $f(f(z)) = z$, therefore the restriction $f|_C : C \to C'$ is a bijection because $f|_{C'} : C' \to C$ is its inverse.

If we assume that Möbius transformations map generalized circles onto generalized circles, we do not need the above arguments. □

(ii) Let C be the circle of center $z_0 \neq 0$ and radius $r \neq z_0$. Show that $f(C)$ is the circle C' of center $w_0 = \frac{\bar{z}_0}{|z_0|^2 - r^2}$ and radius $R = \frac{r}{|z_0|^2 - r^2}$.

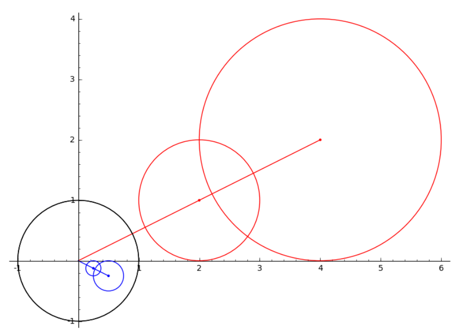

FIGURE 1. The blue circles and the red circles are inverses with respect to the map $z \to \frac{1}{z}$. Red circles have radius 2 and 1 and center $(4, 2)$ and $(2, 1)$ respectively. The biggest red circle 'inverts' to the smallest blue cirlce.

Proof. An arbitrary point $z \in C$ satisfy $|z - z_0| = r$ which implies $r^2 = (z - z_0)\overline{(z - z_0)} = \bar{z}(z - z_0) + (|z_0|^2 - z\bar{z}_0)$. Using this we get

$$\begin{aligned} |f(z) - w_0| &= \left| \frac{1}{z} - \frac{\bar{z}_0}{|z_0|^2 - r^2} \right| \\ &= \left| \frac{|z_0|^2 - r^2 - z\bar{z}_0}{z(|z_0|^2 - r^2)} \right| \\ &= \frac{|\bar{z}| \cdot |z - z_0|}{|z|(|z_0|^2 - r^2)} = R. \end{aligned}$$

The rest follows using the arguments in part (i). □

(iii) Let C be the circle of center $z_0 \neq 0$ and radius $r = |z_0|$. Show that $f(C)$ is the line $\{w : \mathrm{Re}(wz_0) = \frac{1}{2}\} \cup \{\infty\}$.

Proof. An arbitrary point $z \in C \backslash \{0\}$ satisfy $|z - z_0| = |z_0|$, therefore $\bar{z}_0 z_0 = (z - z_0)\overline{(z - z_0)}$ which implies $|z|^2 = z\bar{z}_0 + z_0\bar{z}$. Observe that

$$f(z)z_0 + \overline{f(z)z_0} = \frac{z\bar{z}_0 + \bar{z}z_0}{|z|^2} = 1,$$

which implies $\mathrm{Re}(f(z)z_0) = \frac{1}{2}$. Moreover $z = 0$ goes to ∞, hence $f(z) \in \{w : \mathrm{Re}(wz_0) = \frac{1}{2}\} \cup \{\infty\}$ for $z \in C$. The rest of the argument is same. □

Q5. For $a \in (-1, 1)$, let $D_a = \{z : |z| < 1, \mathrm{Im} z > a\}$. For each such a, either find a Möbius transformation of D_a onto the first quadrant Q, or show that such a transformation cannot exist.

Proof. We will show that there is a Möbius transformation mapping D_a onto first quadrant if and only if $a = 0$.

- **Explicit map when $a = 0$:** Consider the Möbius transformation which sends the points $(0, -1, 1)$ to $(1, 0, \infty)$ which can be explicitly calculated as

  $$f(z) = \frac{1 + z}{1 - z}.$$

  This is in fact equal to $-iC^{-1}$ where $C^{-1} : \Delta(0, 1) \to \mathfrak{h}^+$ is the Cayley transform introduced in class.

  We claim that $f(D_0) = Q$. This comes down to showing that

  $$\mathrm{Im}\ z > 0, \quad |z| < 1 \iff \frac{1 + z}{1 - z} \in Q.$$

  Indeed,

  $$\frac{1 + z}{1 - z} = \frac{(1 + z)(1 - \bar{z})}{|1 - z|^2} = \frac{1 - |z|^2 + (z - \bar{z})}{|1 - z|^2} = \frac{1 - |z|^2 + 2i\mathrm{Im} z}{|1 - z|^2}.$$

  The above expression is in Q iff its real and imaginary parts are positive. This is equivalent to $|z| < 1$ and $\mathrm{Im}\ z > 0$.

- **No such Möbius transformation exist when $a \neq 0$:** Firstly observe that D_a is still intersection of inside of two generalized circles C_1 and C_2. Although when $a \neq 0$ the two circles intersect at angle θ different from $\pi/2$. Let f be a Möbius transformation, hence conformal (respects angles), it sends D_a to the region which is in the intersection of insides of the circles

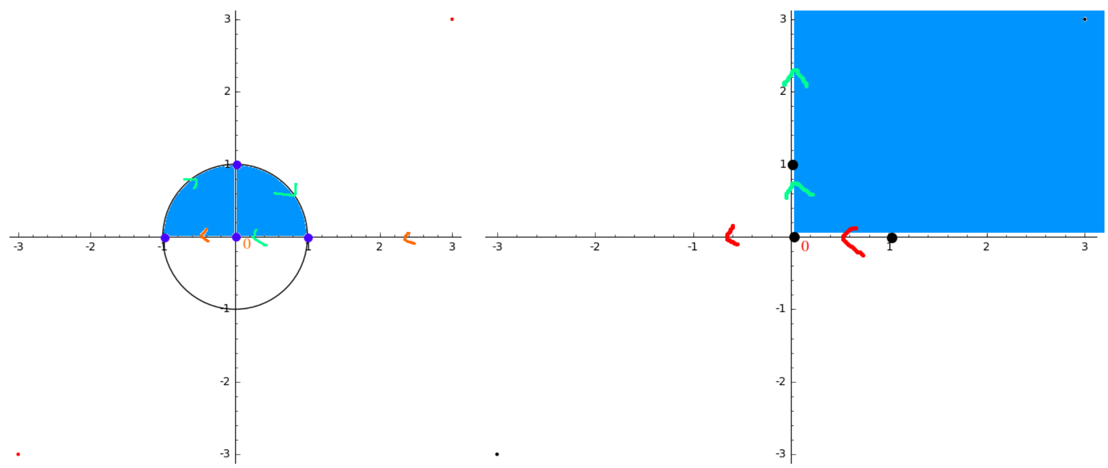

FIGURE 2.

$f(C_1)$ and $f(C_2)$ which intersect at angle θ. Since Q is intersection of inside region of generalized circles ($x = 0$ and $y = 0$) intersection at angle $\pi/2$ while $f(C_1)$ and $f(C_2)$ intersect at angle θ, we conclude $Q \neq f(D_a)$.

□

Q6. Give an example of a biholomorphism between the strip $D = \{z : -\pi < \mathrm{Im} z < \pi\}$ and the slit complex plane $\mathbb{C}^- = \mathbb{C} \backslash \mathbb{R}_{\leq 0}$.

Proof. As the problem suggest, we have a natural map $f : D \to \mathbb{C}^-$ given by $f(z) = e^z$. Note that f is a holomorphic map and $f(D) = \mathbb{C}^-$ since $e^z \in \mathbb{R}_{\leq 0}$ iff $\mathrm{Im} z$ is an odd multiple of π. Moreover the standard branch defined on $\mathbb{C}^-$ is, by construction, is its inverse and it is holomorphic. □

Q7. The arctangent is defined by the power series

$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$$

with radius of convergence $R = 1$. Show that

$$\arctan z = \frac{1}{2i} \mathrm{Log} \frac{1 + zi}{1 - zi}$$

for $z \in \Delta(0, 1)$.

Proof. The radius of convergence of the arctan series is

$$R = \lim_{n \to \infty} \sqrt[n]{2n + 1} = 1,$$

so the left hand side is well defined. We also need to check that the right hand side is well-defined, that is, we need to check that

$$\frac{1 + zi}{1 - zi} \in \mathbb{C}^-.$$

Assume for a contradiction that

$$\frac{1+zi}{1-zi} = -u, \quad u \in \mathbb{R}_{\geq 0} \iff z = i\frac{1+u}{1-u}.$$

Since $|z| < 1$ it follows that

$$|1+u| < |1-u| \iff (1+u)^2 < (1-u)^2 \iff u < 0$$

a contradiction. Next, we see that the first derivative on both sides is $\frac{1}{1+z^2}$:

$$\frac{d}{dz} \arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!} = \frac{1}{1+z^2}$$

$$\frac{d}{dz}\frac{1}{2i}\mathrm{Log}\frac{1+zi}{1-zi} = \frac{1}{2i}\left(\frac{i}{1-zi} + \frac{i}{1+zi}\right) = \frac{1}{1+z^2}.$$

Thus difference of the functions must be a constant. Since the initial values at $z = 0$ agree

$$\arctan 0 = 0 = \frac{1}{2i}\mathrm{Log}1,$$

the given two functions must be the same. □