Math 220A - Fall 2016 - Final Exam

Name: 

Student ID: 

Instructions:

Please print your name and student ID (if you know it).

There are 8 questions which are worth 80 points. You have 180 minutes to complete the test.

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Problem 1. [10 points.]

Consider the function $f(z) = ze^{3-z} - 1$. Show that $f$ has exact one zero inside the disc $\Delta(0, 1)$. 
Problem 2. [10 points.]

Calculate the integral
\[ \int_0^\infty \frac{dx}{x^{2n} + 1}, \text{ for } n \geq 2. \]

Make sure you explain all the necessary estimates.
Problem 3. [10 points.]

Consider

\[ f(z) = z^n + a_1 z^{n-1} + \ldots + a_n. \]

Show that there exists \( c \) with \( |c| = 1 \) such that

\[ |f(c)| \geq 1. \]
Problem 4. [10 points.]

Assume that $f$ is entire and $f(z) = f(z + 1)$ such that $|f(z)| \leq e^{|z|}$. Show that $f$ is constant.

(i) Consider

$$g(z) = \frac{f(z) - f(0)}{\sin \pi z}.$$ 

Show that $g$ is periodic and that $g$ can be extended to an entire function.

(ii) By direct calculation, show that $g$ is bounded in the strip $0 \leq \text{Re } z \leq 1$.

(iii) Conclude from (ii) that $g = 0$ hence $f$ is constant.
Problem 5. [10 points.]

Let \( f(z) = \frac{P(z)}{Q(z)} \) be a rational function with \( \deg P \leq \deg Q - 2 \) such that \( Q \) has no zeros along the non-negative real axis. Show that

\[
\int_0^\infty f(x) \, dx = - \sum_{a \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}} \text{Res}_{z=a}(f(z) \log z),
\]

where for \( z \in \mathbb{C} \setminus \mathbb{R}_{\geq 0} \) we set \( \log z = \log r + i\theta \) and \( \theta \in (0, 2\pi) \).

You may wish to integrate along a “keyhole” contour, consisting of two portions of two circles and two line segments, and avoiding the non-negative real axis.
Problem 6. [10 points.]

Let $a, b \neq 0$ be real numbers and let $U$ be a connected open set. Let $f : U \to \mathbb{C}$ be a holomorphic function. Show that if $a \text{Re} f + b \text{Im} f$ is constant, then $f$ is constant.
Problem 7. [10 points.]

Assume that $f : \mathbb{C} \to \mathbb{C}$ is entire. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$. 
Problem 8. [10 points.]

Assume that $f$ is continuous in the closed unit disc $\Delta$ and holomorphic inside the unit disc $\Delta$. Assume that

$$|f(z)| = 1 \text{ for all } |z| = 1.$$

(i) If $f$ is nonconstant, show that $f$ must have a zero inside $\Delta$.

(ii) Show that if $f$ has a unique simple zero at $z = 0$ then $f(z) = \alpha z$. 