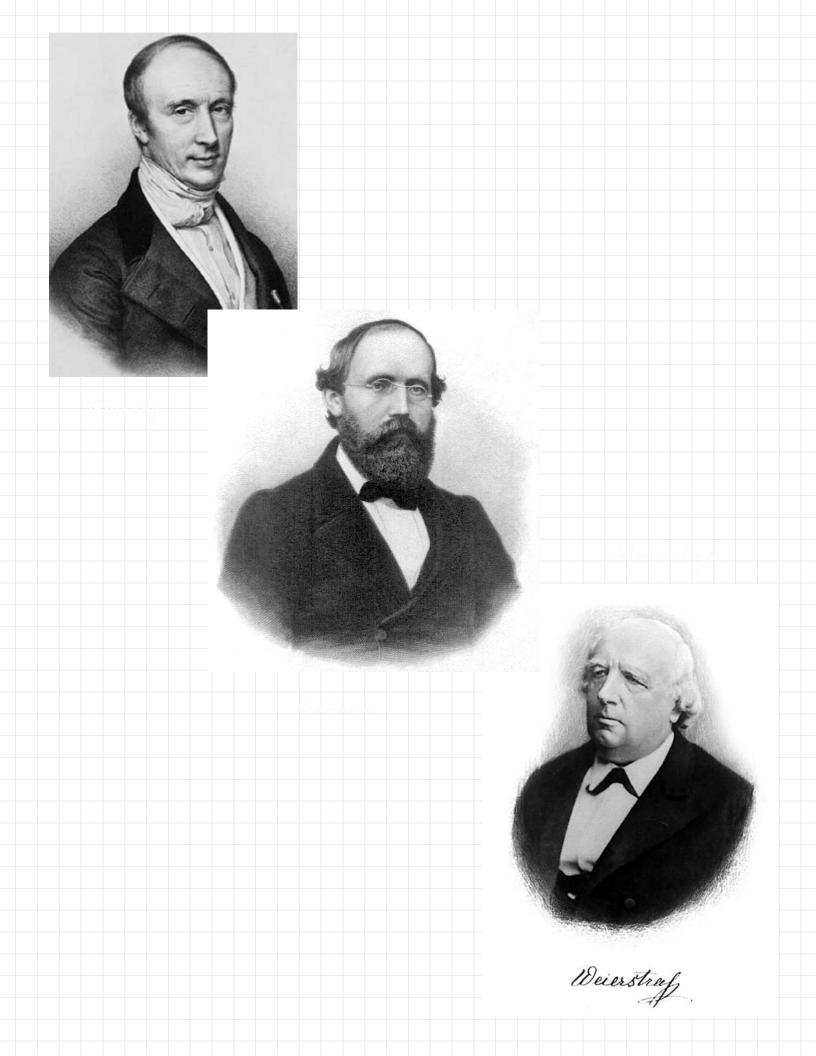
Math 220A - Zecture 1

October 2,2020



Zet u c c open à connected.

Definition f: U - c is complex differentiable (CD) if

 $\lim_{h \to 0} \frac{f(2+k) - f(2)}{k} := f'(2) \text{ exists and is finite.}$ 

Examples 127 f, g complex differentiable => f+g, fg are also

1, 2, 2, 2, ..., 2, ... complex differentiable

2 is not

CD = complex differentiable

RO = real differentiable

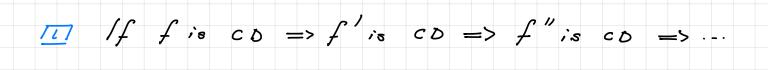


We have seen the same definition for

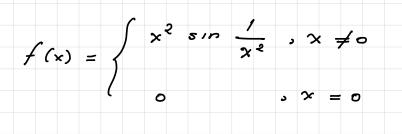
 $f: u \longrightarrow R, u \subseteq R$  open.

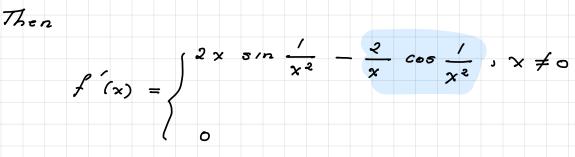
The two definitions have very different

conseguences.



If f is RD this fails. Indeed.





III If f is CD, we will show

 $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n \text{ in some } \Delta(a,r) \leq u.$ 

If fis RD, this fails. Take

 $f(x) = \begin{cases} \tau & -\frac{1}{x^2} \\ 0 & , & x \neq 0 \end{cases}$ 

We have f is (", f(n) (o) = 0 + n so the Taylor

series at 0 is 0. Thus f does not equal its Taylor

series in any interval (-r,r), r>0

Im If f is CD for U = C & f bounded => f constant.

•

If f is RD, f(x) = sinx is bounded.

IV If f is CD and f = o for V = 4 open =>

 $= f \equiv o$ .

This fails if f is RD.

A more appropriate comparison is with functions of two real

Varia bles.

 $|z| = \sqrt{x^2 + y^2}$ 

Definition  $f: U \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$  is real differentiable (RD) if

 $\forall 2 \in \mathcal{U} \quad \exists A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \mathbb{R} = lineor,$ 

We write A = Df(2).

Remark f is CD => f is RD.

Indeed, A: R2 - R2 is multiplication by f(2)

Remark If f = u + iv is RD then

 $u_x, u_y, v_x, v_y = xist and A = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = Jacobian.$ 

Indeed, by definition

 $\lim_{h \to 0} \frac{\left|f(x+h, y) - f(x, y) - h \wedge \left[ \begin{array}{c} \\ \end{array} \right]\right|}{\left|h\right|} = 0$ 

 $=> A \begin{bmatrix} i \\ o \end{bmatrix} = \lim_{h \to o} \frac{f(x+h, y) - f(x, y)}{h} = u_x + iv_x - \sum_{v_x} \begin{bmatrix} u_x \\ v_x \end{bmatrix}$ 

Similarly  $A \begin{bmatrix} 0\\ 1\\ 1\end{bmatrix} = \begin{bmatrix} u_y\\ v_y \end{bmatrix}$ , as needed.

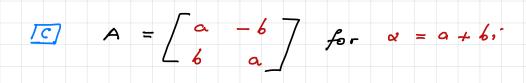
Conversely If ux, uy, v, v, exist & are continuous => f is RD.

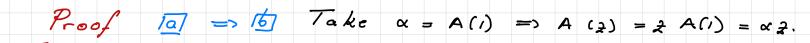
See Moth 140 c or Rudin 9.21.

demma A: IR - /inear. TFAE

A is C - linear

 $[5] A(2) = \sqrt{2} \quad for \quad \alpha \in \sigma$ 





 $\frac{161}{2} \Rightarrow \frac{162}{2} = A\left(\frac{1}{6}\right) = A\left(\frac{1}{6}\right) = \alpha = \begin{bmatrix} a\\b \end{bmatrix}$ 

 $A \begin{bmatrix} 0\\ i \end{bmatrix} = A (i) = a = a = a = b \longrightarrow \begin{bmatrix} -b\\ a \end{bmatrix}$ 

argument above. Thus À is & - linear.

Remark By the demma, TFAE

i fis co

f is RD & Df (2) is C - Intor 42 e U.

Remark (Cauchy - Riemann equations).

 $\begin{array}{cccc} & & & & \\ &$ 

so by the demma

 $u_{x} = v_{y}$ (crequations)

Conversely if CR - equations hold & u, v are of class C'

then  $f = u + i \vartheta$  is  $C \vartheta$ .

Indeed, f is RD in this case and

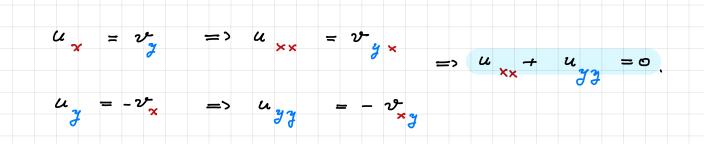
 $Df(z) = \begin{bmatrix} u_x & u_y \\ \vdots & \vdots & \vdots \\ v_x & v_y \end{bmatrix}$  is  $\sigma - linear by the demons$ 

part [] & CR equations. Thus Of (2) is multiplication

by  $\alpha = f'(z) + f is CD$ .

Harmonic functions

If u, & sotiefy CR & are of class C2 then



Similarly v + v = 0 ×× yy

A function to of class C2 with

hxx + hyy = 0 is said to be harmonic.

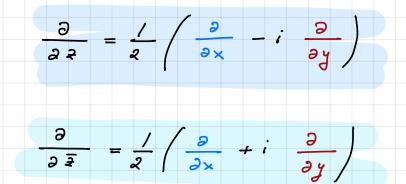
Conclusion

Thus if f is CD & f class C2 then

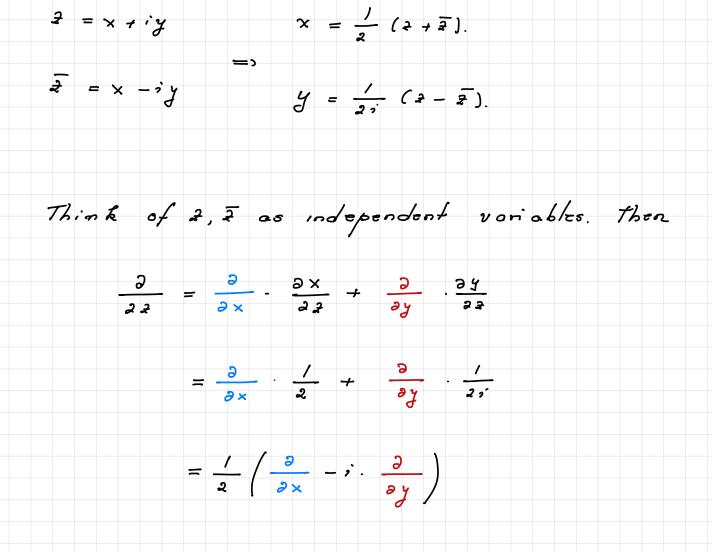
u = Ref, v = Imf are harmonic.

Pairs (u, v) arising this way are called harmonic conjugates.

Notation

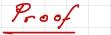


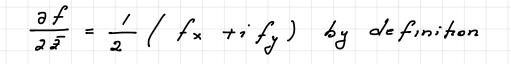


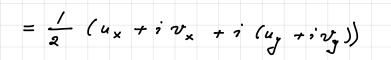


"folepends on 2 but not on 2" Jemma

f is  $cO \Longrightarrow \frac{\partial f}{\partial \overline{x}} = O$ 







 $=\frac{1}{2}(u_{x}-v_{y})+\frac{1}{2}(v_{x}+u_{y})=0$ 

