Math 220A - Lecture 1

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Let $U \subseteq \mathbb{C}$ open & connected.

Definition $f : U \longrightarrow \mathbb{C}$ is complex differentiable (CD) if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} := f'(z) \text{ exists and is finite.}$$

Examples

[I] f, g complex differentiable $\Rightarrow f+g, fg$ are also

[II] $1, z, z^2, \dots, z^n, \dots$ complex differentiable

$\bar{z}$ is not

CD = complex differentiable

RD = real differentiable

Remark

We have seen the same definition for

$f : U \longrightarrow \mathbb{R}$, $U \subseteq \mathbb{R}$ open.

The two definitions have very different consequences.

$\boxed{1}$ If f is CD $\Rightarrow$ f' is CD $\Rightarrow$ f'' is CD $\Rightarrow \dots$

If f is RD this fails. Indeed,

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x^2} &, x \neq 0 \\ 0 &, x = 0 \end{cases}$$

Then

$$f'(x) = \begin{cases} 2x \sin \dfrac{1}{x^2} - \dfrac{2}{x} \cos \dfrac{1}{x^2} &, x \neq 0 \\ 0 & \end{cases}$$

is not even continuous.

[II] If f is CD, we will show

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n \quad \text{in some } \Delta(a,r) \subseteq U.$$

If f is RD, this fails. Take

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

We have f is C^∞, $f^{(n)}(0) = 0$ $\forall n$ so the Taylor series at 0 is 0. Thus f does not equal its Taylor series in any interval $(-r, r)$, $r > 0$.

[III] If f is CD for $U = \mathbb{C}$ & f bounded $\Rightarrow$ f constant.

If f is RD, $f(x) = \sin x$ is bounded.

[IV] If f is CD and $f = 0$ for $V \subseteq U$ open $\Rightarrow$

$\Rightarrow f \equiv 0$.

This fails if f is RD.

A more appropriate comparison is with functions of two real variables.

Identify $\mathbb{C} \cong \mathbb{R}^2$; $z = x + iy \longleftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

$$|z| = \sqrt{x^2 + y^2}$$

Definition $f : U \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is real differentiable (RD) if $\forall z \in U$ $\exists$ $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\mathbb{R}$ – linear,

$$\lim_{h \to 0} \frac{|f(z+h) - f(z) - Ah|}{|h|} = 0.$$

We write $A = Df(z)$.

Remark f is CD $\Rightarrow$ f is RD.

Indeed, $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is multiplication by $f'(z)$

Remark If $f = u + iv$ is RD then

u_x, u_y, v_x, v_y exist and $A = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ = Jacobian.

Indeed, by definition

$$\lim_{h \to 0} \frac{\left| f(x+h, y) - f(x,y) - h\, A \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right|}{|h|} = 0$$

$$\Rightarrow A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h} = u_x + i v_x \rightsquigarrow \begin{bmatrix} u_x \\ v_x \end{bmatrix}$$

Similarly $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} u_y \\ v_y \end{bmatrix}$, as needed.

Conversely If

u_x, u_y, v_x, v_y exist & are continuous $\Rightarrow f$ is RD.

See Math 140 C or Rudin 9.21.

Lemma $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. $\mathbb{R}$ - linear. TFAE

$\boxed{a}$ A is $\mathbb{C}$ - linear

$\boxed{b}$ $A(z) = \alpha z$ for $\alpha \in \mathbb{C}$

$\boxed{c}$ $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for $\alpha = a + bi$

Proof $\boxed{a} \Longrightarrow \boxed{b}$ Take $\alpha = A(1) \Longrightarrow A(z) = z\,A(1) = \alpha z$.

$\boxed{b} \Longrightarrow \boxed{c}$ $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A(1) = \alpha = \begin{bmatrix} a \\ b \end{bmatrix}$

$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A(i) = \alpha i = ai - b \longrightarrow \begin{bmatrix} -b \\ a \end{bmatrix}$

$\boxed{c} \Longrightarrow \boxed{a}$ Let $\alpha = a + bi$. Then $A(z) = \alpha z$ by the argument above. Thus A is $\mathbb{C}$ - linear.

Remark By the Lemma, TFAE

[i] f is CD

[ii] f is RD & $Df(z)$ is $\mathbb{C}$-linear $\forall z \in \mathcal{U}$.

Remark (Cauchy-Riemann equations).

If f is CD, $Df(z) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ is $\mathbb{C}$-linear

so by the Lemma

$$u_x = v_y$$

$$u_y = -v_x$$

(CR equations)

Conversely if CR-equations hold & u, v are of class C^1

then $f = u + iv$ is CD.

Indeed, f is RD in this case and

$$Df(z) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$ is $\mathbb{C}$-linear by the Lemma

part $\boxed{c}$ & CR equations. Thus $Df(z)$ is multiplication

by $\alpha = f'(z)$ & f is CD.

Harmonic functions

If u, v satisfy CR & are of class C^2 then

$$u_x = v_y \implies u_{xx} = v_{yx}$$
$$u_y = -v_x \implies u_{yy} = -v_{xy}$$
$$\implies u_{xx} + u_{yy} = 0.$$

Similarly $v_{xx} + v_{yy} = 0$

A function h of class C^2 with

$h_{xx} + h_{yy} = 0$ is said to be harmonic.

Conclusion

Thus if f is CD & f class C^2 then

$u = \mathrm{Re}\, f$, $v = \mathrm{Im}\, f$ are harmonic.

Pairs (u, v) arising this way are called harmonic conjugates.

Notation

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)$$

Remark

$$\begin{aligned} z &= x + iy \\ \bar{z} &= x - iy \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x &= \frac{1}{2}(z + \bar{z}). \\ y &= \frac{1}{2i}(z - \bar{z}). \end{aligned}$$

Think of $z, \bar{z}$ as independent variables. Then

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial z}$$

$$= \frac{\partial}{\partial x} \cdot \frac{1}{2} + \frac{\partial}{\partial y} \cdot \frac{1}{2i}$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} - i \cdot \frac{\partial}{\partial y}\right)$$

Lemma

"f depends on z but not on $\bar{z}$"

f is CD $\Longrightarrow \dfrac{\partial f}{\partial \bar{z}} = 0$

Proof

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(f_x + i f_y \right) \text{ by definition}$$

$$= \frac{1}{2}\left(u_x + i v_x + i (u_y + i v_y) \right)$$

$$= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y) \stackrel{?}{=} 0$$

$$\Longleftrightarrow \quad \begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

. These are the CR equations.