Theorem \( f: U \to \mathbb{C} \) holomorphic, non constant \( \Rightarrow \) \nolimits

1. \( f \) cannot have local maxima.

Proof: Assume that \( f \) achieves a local maximum at \( a \).

\[ \Rightarrow \exists \ V \ni a, \ v \in U, \ \text{if} f \text{ has a maximum at } a. \]

By OMT, \( f(v) \) is open. \( \Rightarrow \exists \) disc \( \Delta \) centered at \( f(a) \)

\( \Delta \subseteq f(v) \). Note that \( f \) measures distance from the origin. The disc \( \Delta \) has points farther from \( 0 \) than \( f(a) \).

contradicting the assumption \( f \) has maximum at \( a \). (in \( v \)).
Remarks

Minimum modulus principle

\[ f: \mathbb{U} \rightarrow \mathbb{C} \text{ holomorphic, not constant, } f \text{ has no zeros in } \mathbb{U}. \]

\[ \Rightarrow \text{ } \|f\| \text{ has no local minimum} \]

Proof. Let \( g = \frac{1}{f}: \mathbb{U} \rightarrow \mathbb{C} \text{ holomorphic}. \) Apply the maximum modulus to the function \( g \) and conclude.

\[ \text{If } I \text{ bounded, } f: \mathbb{U} \rightarrow \mathbb{C} \text{ continuous, holomorphic in } \mathbb{U} \]

\[ \Rightarrow \max_{\mathbb{U}} |f| = \max_{\mathbb{U}} |f| \text{ (*)}. \]

Proof. Since \( \mathbb{U} \text{ bounded } \Rightarrow \mathbb{U}, \text{ in } \mathbb{U} \text{ compact so } f \text{ achieves maxima on these sets. Let } f \text{ achieve maximum in } \mathbb{U} \text{ at } \]

\[ a \in \mathbb{U}. \]

If \( a \in \mathbb{U} \Rightarrow f|_a \text{ has a maximum at } a \Rightarrow \]

\[ \Rightarrow f = \text{constant} \text{ & there's nothing to prove.} \]

Otherwise \( a \in \mathbb{U} \) proving (*).
We have seen that if $f : \Delta(a, r) \rightarrow \mathbb{C}$ then

$$f(z) = \sum_{k=0}^{\infty} a_k (z - a)^k$$

We consider Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - a)^k$$

**Convergence of Laurent series**

$$f^+(z) = \sum_{k=0}^{\infty} a_k (z - a)^k$$

$$f^-(z) = \sum_{k=-\infty}^{-1} a_k (z - a)^k = \sum_{k=1}^{\infty} a_k (z - a)^{-k}$$

$$f(z) = f^+(z) + f^-(z)$$

**Definition**

$f$ converges absolutely and uniformly provided $f^+, f^-$ do so.

**Remark**

The radius of convergence

$f^+$ converges if $|z - a| < R$.

$f^-$ converges if $|z - a| < r' \iff |z - a| > r$. 

**Laurent Series & Functions in annular regions (Conway 5.1)**
For power series, convergence is **absolute** & **uniform** on compact subsets.

\[ D = \text{finite} \quad \Delta (a; r, R) = \{ z : r < |z - a| < R \} , \quad 0 \leq r < R \leq \infty. \]

**Theorem** Let \( f : \Delta (a; r, R) \rightarrow \mathbb{C} \) be holomorphic. Then \( f(z) = \sum_{k=-\infty}^{\infty} a_k (z - a)^k \) can be expanded into a **Laurent series**, converging absolutely & uniformly on compact sets in \( \Delta (a; r, R) \). Furthermore,

\[ a_k = \frac{1}{2\pi i} \int_{|w - a| = \rho} \frac{f(w)}{(w - a)^{k+1}} \, dw , \quad r < \rho < R. \]

**Remark** An important case is \( r = 0 \). Then

\[ \Delta^* (0, R) = \Delta (0, R) \setminus \{ 0 \} = \text{punctured disc}. \]

Let \( f : \Delta^* (0, R) \rightarrow \mathbb{C} \) be holomorphic \( \Rightarrow f(z) = \sum_{k=-\infty}^{\infty} a_k (z - 0)^k \)

Compare this to **Taylor expansion**.


Pierre Alphonse Laurent
1813 - 1854
(engineer in the army).

The original work on Laurent series was not published.

Cauchy writes:

"L'extension donnée par M. Laurent... nous paraît
digne de remarque."
Proof (of Laurent expansion) \( A = \Delta(a; r, R) \).

\[ \text{wlog } a = 0; \text{ else work with } f(z + a). \]

the expression \( a_k = \frac{1}{2\pi i} \int_{\gamma_p} \frac{f(z)}{z^{k+1}} \, dz \)

is independent of \( p \). Indeed

\( \gamma_p \sim \gamma_p' \) and use

Cauchy Homotopy Theorem.

suffices to prove pointwise convergence. m2:

Indeed, convergence of \( f \) \( \Rightarrow \) convergence of \( f^+ \) & \( f^- \) in \( r < 121 < R \).

But then \( f^+ \) converges in \( 121 < R \) (power series have discs of convergence) & we remarked convergence is absolute & uniform on compacts. Same for \( f^- \).
Pointwise convergence

Let \( r < s < 2 \) \( \leq S < R \)

Let \( S \) be a segment joining \( Y_s, Y_S \)

avoiding \( z \).

Let

\[ \gamma = Y_S + S + Y_a + (-S) \]

Note \( \gamma \sim 0 \). This can be seen by continuously shrinking \( S \to 0 \).

Also \( n(\gamma, z) = 1 \) since \( n(Y_s, z) = 0 \) as \( z \) is outside and
\( n(Y_S, z) = 1 \) as \( z \) is interior to \( Y_S \). \( \Rightarrow n(\gamma, z) = 1 \).

CIF:

\[ (+) \quad f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} \, dw. \]

\[
= \frac{1}{2\pi i} \int_{\gamma_S} \frac{f(w)}{w-z} \, dw - \frac{1}{2\pi i} \int_{Y_S} \frac{f(w)}{w-z} \, dw
\]

(cancelling the contribution of \( \delta, -\delta \)).

The two terms will give the positive/inegative parts
of Laurent series.
Key expansions (Remember them) \( \alpha < 1 \Rightarrow \alpha < s. \)

\[
\frac{1}{w - 2} = \frac{1}{w} \cdot \frac{1}{1 - \frac{2}{w}} = \sum_{k=0}^{\infty} \frac{1}{w} \left( \frac{2}{w} \right)^k
\]

\[
= \sum_{k=0}^{\infty} \frac{2^k}{w^{k+1}}. \tag{1}
\]

The convergence is uniform in \( w \) since \( \left| \frac{2}{w} \right| = \frac{12}{S} < 1 \). We can define \( M_k = \frac{12^k}{S^{k+1}}, f_k(z) = \frac{2^k}{w^{k+1}} \) and invoke Weierstrass M-test to conclude uniform convergence.

We can multiply by \( f(w) \). Uniform convergence still holds. (Use \( M_k = \frac{12^k}{S^{k+1}}, \sup_{z} |f(z)|. \))

We can then integrate term by term. (Rudin). Thus

\[
\frac{1}{2\pi i} \int_{\gamma_5} \frac{f(w)}{w - 2} \, dw = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\gamma_5} \frac{f(w)}{w^{k+1}} \, dw \cdot 2^k
\]

\[
= \sum_{k=0}^{\infty} c_k 2^k. \tag{2}
\]

(ii) over \( \gamma_3 \), we use a different expansion

\[
\frac{1}{w - 2} = -\frac{1}{2} \cdot \frac{1}{1 - \frac{2}{w}} = \sum_{k=0}^{\infty} -\frac{1}{2} \left( \frac{w}{2} \right)^k
\]

\[
= \sum_{k=0}^{\infty} -\frac{w^k}{2^{k+1}}. \tag{3}
\]
Here \( \left| \frac{w}{z} \right| = \frac{3}{|z|} < 1 \). By the same arguments

\[
-\frac{1}{2\pi i} \int_{\gamma_3} \frac{f(w)}{w-z} \, dw \quad \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\gamma_3} f(w) w^k \, dw \cdot z^{-k-1}
\]

\[
= \sum_{k=0}^{\infty} a_{-k-1} z^{-k-1}
\]

\[
= \sum_{k=-\infty}^{\infty} a_k z^k \quad (**) \]

(\+), (*) \quad (**). imply the Theorem.