Last time we wish to prove:

**Residue Theorem** \( U \) is an open connected, \( S \) discrete

1. \( \gamma \sim 0, \{ \gamma \} \subseteq U \setminus S. \)

2. \( f \) holomorphic in \( U \setminus S \), singularities at \( S \).

Then

\[
\frac{1}{2\pi i} \oint_{\gamma} f \, dz = \sum_{s \in S} \text{Res} \, (f, s) \cdot n(\gamma, s).
\]
Example

\[ \int \frac{2 + 1}{z^2 (z-1)} \, dz \]

\[ \text{Res} \left( \frac{2 + 1}{z^2 (z-1)} \right) \]

Take \( u = \Delta (0,1) \), \( S = \{0,1\} \), \( f(z) = \frac{2 + 1}{z^2 (z-1)} \).

- \( \text{Res} \left( f, 0 \right) = \text{Res} \left( \frac{2 + 1}{z^2 (z-1)} \right) \)
  \[ \frac{2 + 1}{z^2} \]
  \[ \left. \frac{(2 + 1)}{z^2} \right|_{z=0} = -2 \]
  by Method 2 of computing residues last time.

- \( \text{Res} \left( f, 1 \right) = \text{Res} \left( \frac{2 + 1}{z^2 (z-1)} \right) \)
  \[ \frac{2 + 1}{z^2 (z-1)} \]
  \[ \left. \frac{(2 + 1)}{z^2 (2-1)} \right|_{z=1} = \frac{2}{(2^2 (2-1))} = \frac{2}{1} = 2 \]
  by Method 1 of computing residues last time.

Thus

\[ \int \frac{f \, dz}{2 + 1} = -2 \pi i \left( \text{Res} \left( f, 0 \right) + \text{Res} \left( f, 1 \right) \right) \]

\[ = 2 \pi i \left( -2 + 2 \right) = 0. \]
1. Proof of the Residue Theorem

Terminology

$U^* \subseteq \mathbb{C}$, $\gamma^* = \sum_{i=1}^{l} m_i \gamma_i$, $C'$-chain

\[ \int_{\gamma^*} f \, dz = \sum_{i=1}^{l} m_i \int_{\gamma_i} f \, dz \]

\[ n(\gamma^*, a) = \sum_{i=1}^{l} m_i n(\gamma_i, a) \]

Definition

$\gamma^* \sim 0$ if $n(\gamma^*, a) = 0 \quad \forall \ a \not\in U^*$.

(we say $\gamma^*$ is null homologous in $U^*$).

Remark

If $\gamma^*$ loop in $U^*$. Then

$\gamma^* \sim 0 \implies \gamma^* \sim 0$.

Indeed if $a \not\in U^*$ then

$\left. \frac{d}{dw} \right|_{w=a} = 0$

by homotopy form of Cauchy, applied to $\gamma^* \sim 0$ and to the holomorphic function $\frac{1}{w-a}$ in $U^*$, $(a \not\in U^*)$.
the converse is false \( U = \mathbb{T} \setminus \{a, b\} \)

Check \( \gamma^* \cong 0 \). Indeed \( n(\gamma^*, a) = n(\gamma^*, b) = 0 \). To see this, find two subloops of \( \gamma^* \) going clockwise & counter-clockwise once around \( a \) (same for \( b \)). However \( \gamma^* \not\cong 0 \).

Remark In algebraic topology, one learns that 1st homology is the abelianization of \( \pi_1 \) (which is defined via homotopy).
Enhanced Cauchy's Theorem

We seek to prove a "homology" version of Cauchy:

**Theorem** \( f \colon U^* \to \mathbb{C} \) holomorphic, \( \gamma^* \simeq 0 \) \( \Rightarrow \int f \, dz = 0 \).\( \gamma^* \)

Of course, this implies the homotopy version of the theorem, proved in previous lectures.

Remark We show next

Enhanced Cauchy Theorem \( \Rightarrow \) Residue Theorem.
Proof of residue theorem

We let $f$ holomorphic in $U \setminus S$, $\gamma \sim 0$. We want

$$\frac{1}{2\pi i} \int_{\gamma} f \, dz = \sum_{j \in S} \text{Res} (f, s) \cdot n (\gamma, s).$$

Last time we saw RHS is finite since

$$\left\{ j \in S : n (\gamma, s) \neq 0 \right\}$$

is finite.

Enumerate this set to be $\left\{ a_1, \ldots, a_k \right\}$, $m_i = n (\gamma, a_i) \neq 0$.

Let $\Delta_j$ be small disjoint discs near $a_i$, $\Delta_j \subseteq U$.

Define $U^* = U \setminus S$.

$$\gamma^* = \gamma + \sum_{i=1}^{k} (-m_i) C_i,$$

where $C_i = \partial \Delta_j$.

(positive orientation)

Claim $\gamma^* \sim 0$

Enhanced Cauchy for $(U^*, \gamma^*)$ $\Rightarrow \int_{\gamma^*} f \, dz = 0$

$$\Rightarrow \frac{1}{2\pi i} \int_{\gamma} f \, dz = \sum_{i=1}^{k} m_i \cdot \frac{1}{2\pi i} \int_{C_i} f \, dz$$

$$= \sum_{i=1}^{k} m_i \cdot \text{Res} (f, a_i)$$

by toy example last time. QED.
Proof of the claim. \[ \text{Want } n(\gamma^*, a) = 0 \text{ if } a \not\in U^*. \]

\[ \text{if } a \not\in U. \text{ Note } u \text{ if } a \not\in U. \text{ Note } u \text{ if } u \Rightarrow u \Rightarrow n(\gamma, a) = 0. \]

Also \[ a \not\in \Delta \Rightarrow n(c_i, a) = 0 \] Then

\[ n(\gamma^*, a) = n(\gamma, a) + \sum (-m_i) n(c_i, a) = 0. \]

\[ \left\{ \begin{array}{ll} 0 & \text{if } a \neq a_i; \\ 1 & \text{if } a = a_i; \end{array} \right. \]

If \[ a = a_i \Rightarrow n(\gamma^*, a) = n(\gamma, a) + (-m_i) n(c_i, a) = m_i + (-m_i) = 0. \]

If \[ a \neq a_i \Rightarrow n(\gamma, a) = 0 \text{ by definition of the } a_i's \]

\[ \Rightarrow n(\gamma^*, a) = n(\gamma, a) + \sum (-m_i) n(c_i, a) = 0. \]
Remarks

[17] Proof of residue thm only requires \( u \neq 0 \), not \( u \neq 0 \). Improvement of hypothesis.


Let \( u \neq 0 \). Apply the residue theorem: \( s = \{a\} \).

\[
\frac{1}{2\pi i} \int \frac{f(z)}{z} \frac{dz}{(z - a)^{k+1}} = n(y, a) \text{ Res}_{z=a} \frac{f(z)}{(z-a)^{k+1}}
\]

\[
= n(y, a) \cdot \frac{f^{(k)}(a)}{k!}
\]

(using Method 2 from last time)
2. **Proof of Enhanced Cauchy's Theorem**

- change notation \( u \leftarrow u^*, \gamma \leftarrow \gamma^* \)

- modify statement slightly

**Theorem (enhanced CIF)**

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} \, dz = n(\gamma, a) \cdot f(a).
\]

**Remark** Using the above for \( f^{(m)}(a) = f(z), (z-a), f^{(m)}(a) = 0 \)

we obtain \( \gamma \approx 0 \Rightarrow \int f \, dz = 0 \). This is **Enhanced Cauchy**

**Remark** TFAE:

Enhanced CIF \( \Rightarrow \) Enhanced Cauchy's Theorem

\( \Rightarrow \) Residue Theorem

\( \Rightarrow \) Enhanced CIF for derivatives
Next time we give Dixon's proof of Enhanced CIF.