Last time: **Winding number (index)**

\[ n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a} \in \mathbb{Z} \]

\[ n(\gamma, a) = 1. \]

\[ n(\gamma, a) = 2 \]

**Properties**

1. \( n(-\gamma, a) = -n(\gamma, a) \) (change of orientation)

**Proof:**

\[ \int_{-\gamma} \frac{dz}{z - a} = -\int_{\gamma} \frac{dz}{z - a} \]
\[ n(\gamma, \cdot) \colon \mathbb{C} \setminus \{\gamma\} \to \mathbb{Z} \text{ is locally constant} \]

\[ n(\gamma, a) = 0 \text{ for } a \text{ in the unbounded component of } \mathbb{C} \setminus \{\gamma\}. \]

**Proof.**

Let \( R \) be a component of \( \mathbb{C} \setminus \{\gamma\} \). If \( a, b \in R \Rightarrow a, b \) can be joined by a polygonal path in \( R \).

This is the same argument used in the past to show we can join by piecewise \( C^1 \) path. Suffices to show

if \( \overline{ab} \subseteq R \Rightarrow n(\gamma, a) = n(\gamma, b) \)

\[ \iff \int_\gamma d\zeta \left( \frac{1}{2-a} - \frac{1}{2-b} \right) = 0. \]

This is true since \( \log \frac{2-a}{2-b} \) is a primitive of the
integrand. We showed last time \( \log \frac{2-a}{2-b} \) is well defined in \( \mathbb{C} \setminus \overline{ab} \).

If \( U \) is the unbounded component, let \( R > 2 \alpha \) such that \( \{ \gamma \} \subseteq \Delta(0,R) \). Let \( m \) be the value of \( n(\gamma,-) \) on \( U \). Pick \( 1\alpha \geq 2R \).

\[ a \in U \implies \left| \frac{2-a}{\alpha} \right| \geq \frac{1}{|\alpha|} \left| 1-\frac{1}{2} \right| \geq 2R - R = R \] if \( \gamma \in \{ \gamma \} \implies \)
\[ |m| = |n(\gamma,a)| = \frac{1}{2\pi} \left| \int_{\gamma} \frac{dz}{2-a} \right| \leq \frac{1}{2\pi} \frac{1}{R} \text{ length}(\gamma). \]

Make \( R \to \infty \implies n(\gamma,a) = m = 0. \)
\[ \gamma = \gamma_1 + \gamma_2 \]

\[ \Rightarrow \pi (\gamma, a) = \pi (\gamma_1, a) + \pi (\gamma_2, a) \]

Proof:

\[ \int_{\gamma} \frac{d\zeta}{\zeta - a} = \int_{\gamma_1} \frac{d\zeta}{\zeta - a} + \int_{\gamma_2} \frac{d\zeta}{\zeta - a}. \]
Two questions arise

(a) Can we define integrals over γ continuous?

(b) \( \gamma_1 \sim \gamma_2 \Rightarrow \pi_1(X, a) = n(\gamma_1, a) = n(\gamma_2, a) \).

Answer to (a) YES. If \( f \) holomorphic, \( \gamma \) continuous we define \( \int_{\gamma} f \). For instance by analytic continuation.

We will not pursue this here.

Answer to (b) YES. Cauchy’s Theorem (Homotopy)
We reparametrize so that the domain is $I = [0,1]$.

Homotopy $\gamma_0, \gamma_1: I \rightarrow U$ continuous loops

$\gamma_0 \sim \gamma_1$ if $\exists h: I \times I \rightarrow U$ continuous

\[ h(t, 0) = \gamma_0(t), \quad h(t, 1) = \gamma_1(t). \]

\[ h(0, s) = h(1, s). \]

$\Rightarrow \quad \gamma_s(t) = h(t, s).$ continuous loop.
**Def** \( \gamma_0, \gamma_1 : I \rightarrow U \) continuous paths from \( p \) to \( q \)

\[ \gamma_0 \sim \gamma_1 \text{ if } \exists h : I \times I \rightarrow U \text{ continuous} \]

\[ h(t, 0) = \gamma_0(t), \quad h(t, 1) = \gamma_1(t). \]

\[ h(0, s) = p, \quad h(1, s) = q. \]
Remark 1a) \( \sim \) is an equivalence relation.

\[
\gamma_0 \sim \gamma_1, \ \gamma_1 \sim \gamma_2 \implies \gamma_0 \sim \gamma_2
\]

[\text{Check}] \ \gamma + (-\gamma) \sim 0. \ \text{& \ \gamma \text{ path in } U}

[c] If \( \gamma \in \mathbb{F} \), let \( \gamma = \gamma_0 + (-\gamma_1) \) loop

\[
\implies \gamma \sim 0. \ \text{as loops. Indeed let}
\]

\[
\Gamma_s = \gamma_s + (-\gamma_1).
\]

\[
\Gamma_0 = \gamma. \ \text{By \text{[c]}}, \ \Gamma_1 \sim 0.
\]

\[
\text{By \text{[c]}} \implies \gamma \sim 0.
\]

Def \( U \) is simply connected if \( \forall \gamma \text{ loop in } U, \)

\[
\gamma \sim 0 \iff \overline{\pi_1}(u) = 0.
\]
Example

\( U \) is star convex \( \Rightarrow \) \( U \) simply connected

**Def** \( U \) star convex if \( \exists \ p_0 \in U \) such that \( \forall \ p \in U \Rightarrow \overline{p_0 \ p} \subseteq U \).

Let \( \gamma \) be a loop in \( U \).

\[ L(t, s) = s \ p_0 + (1-s) \ \gamma(t) \subseteq U \]

\[ L(t, 0) = \gamma(t) \]

\[ L(t, 1) = \ p_0 \Rightarrow \gamma \sim 0. \]
**Cauchy's Theorem** (Homotopy version)

\[ f : U \rightarrow \mathbb{C} \text{ holomorphic, } \gamma_0 \sim \gamma_1 \text{ piecewise } \]

\[ C' \text{ loops in } U \implies \int_{\gamma_0} f \, dz = \int_{\gamma_1} f \, dz. \]

**Remarks**

\[ \gamma \sim 0 \implies \int_{\gamma} f \, dz = 0. \]

If \( U \) simply connected \( \implies \int_{\gamma} f \, dz = 0 \forall \gamma \text{ C'} \text{ loop in } U. \)

\[ \gamma_1, \gamma_2 \text{ piecewise } C' \text{ paths, } \gamma_1 \sim \gamma_2 \]

\[ \implies \int_{\gamma_1} f \, dz = \int_{\gamma_2} f \, dz. \text{ Indeed let } \gamma = \gamma_1 + (-\gamma_2). \]

By \[ \gamma \implies \int_{\gamma} f \, dz = 0 \implies \int_{\gamma_1} f \, dz = \int_{\gamma_2} f \, dz. \]

\[ \gamma_0 \sim \gamma_1, U \subseteq \bigcup \{ \gamma \} \text{ piecewise C' loops in } U \subseteq \mathcal{C} \]

\[ \implies \int_{\gamma_0} \frac{d\zeta}{2-\zeta} = \int_{\gamma_1} \frac{d\zeta}{2-\zeta} \]

\[ \implies \nu(\gamma_0, a) = \nu(\gamma_1, a). \]

This proves a previous assertion.
Remark: The homotopy in Cauchy's theorem is not assumed to be $C^1$.

Existence of primitives in simply connected sets

If $U$ simply connected, $f: U \to \mathbb{C}$ holomorphic

$$\Rightarrow \int_C f(z) dz = 0.$$ by Remark II

$$\Rightarrow \text{Prop A, } f \text{ has a primitive}$$

Corollary: Any holomorphic function in a simply connected set admits a primitive.

Take $f(z) = \frac{1}{z}$. A primitive is a branch of logarithm.

Corollary: Let $U \subseteq \mathbb{C} \setminus \{0\}$ simply connected. We can define a branch of logarithm in $U$. 