Math 220, Practice problems for the midterm.

Please review the homework questions in addition to these practice problems.

1. Let \( f : U \to \mathbb{C} \) be a holomorphic function defined in a connected open set \( U \). Assume that for each \( z \in U \), there are positive integers \( m \) and \( n \) (that may depend on \( z \)) such that
   \[
   f(z)^m = \overline{f(z)}^n.
   \]
   Show that \( f \) is a constant.

2. (You could solve this problem on Monday.) Find the Laurent expansions around 0 for the function
   \[
   f(z) = \frac{1}{z^2 + 5z + 4}
   \]
   valid in three different regions of the complex plane.

3. Using Cauchy’s integral formula, calculate the following integrals:
   (i) \[
   \int_{|z-1|=1} \frac{\sin(\pi z)}{(z^2 - 1)^2} \, dz
   \]
   (ii) \[
   \int_{|z-1|=a} \frac{e^z}{z^2 - 2z} \, dz.
   \]

5. Let \( \Delta \) be the open unit disc. Let \( f : \overline{\Delta} \to \mathbb{C} \) be a nonconstant continuous function on the closed unit disc, holomorphic on the open disc \( \Delta \). Assume that \( f(\partial \Delta) \subset \partial \Delta \).
   (i) Show that \( f(\Delta) \subset \Delta \).
   (ii) Show that \( f \) must have a zero inside \( \Delta \).

6. (We should cover the material for this on Wednesday.)
   (i) Find the residue at \( z = -1 - i \) for the function
      \[
      f(z) = \frac{z \log(z)}{(z + 1 + i)^2}.
      \]
      Here, the principal branch of the logarithm is used.
   (ii) For what value of \( a \), does the function
      \[
      \frac{1}{e^z - 1} + \frac{a}{\sin z}
      \]
      has a removable singularity at the origin?

7. Assume that \( f : \overline{\Delta} \to \mathbb{C} \) is a continuous function defined on a closed disc \(|z| \leq r\) and holomorphic inside the disc \(|z| < r\). Prove that Cauchy’s integral formula
   \[
   f(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta - z} \, d\zeta
   \]
holds for $|z| < r$.

*Hint: This requires an argument; the usual Cauchy integral formula stated in class does not apply directly. Instead use circles of radii $r - \frac{1}{n}$ and let $n \to \infty$.*

8. *(We should cover the material for this on Wednesday. Not required for the midterm.*) Prove the Casorati-Weierstrass theorem: if $f : \Delta \setminus \{0\} \to \mathbb{C}$ is a holomorphic function on the punctured unit disc with an essential singularity at the origin, then $f(\Delta \setminus \{0\})$ is dense in $\mathbb{C}$.

9. Let $U$ be open and connected, and let $f, g$ be holomorphic functions such that $f(z)g(z) = 0$. Show that either $f$ or $g$ is identically zero on $U$.

10. *(We should cover the material for this on Wednesday. Not required for the midterm.*) Show that there is no meromorphic function $f$ on the unit disc $\Delta(0, 1)$ such that $f'$ has a pole of order exactly one at $z = 0$.

11. Consider the holomorphic function

$$f(z) = e^z + ie^{-z}$$

over the closed rectangle $R$ with corners

$$\pm 1 \pm i\frac{\pi}{2}.$$ 

Find the maximum of $f$ and confirm that it lies over the boundary of $R$. Where does the minimum occur?

12. Show that a function $f : \mathbb{C} \to \mathbb{C}$ which is entire and doubly periodic must be constant. A function $f$ is doubly periodic provided

$$f(z) = f(z + \omega_1) = f(z + \omega_2)$$

for complex numbers $\omega_1, \omega_2$ such that $\omega_1/\omega_2 \notin \mathbb{R}$.

13. Let $f$ be an entire function such that $|f(z)| \leq e^{\text{Re}z}$. Show that either $f = 0$ or else $f$ has no zeros in $\mathbb{C}$.

14. Suppose that $f$ is entire and $\frac{f(z)}{1+|z|^{1/2}}$ is bounded. Prove that $f$ is constant.