Math 220A - Fall 2020 - Midterm

Name: _____________________________________

Student ID: ________________________________

Instructions:

Please print your name and student ID (if you know it).

You may not use any books, notes or internet.

There are 4 questions which are worth 40 points. You have 60 minutes to complete the test. Please upload your answers in Gradescope at the end of the exam.

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Problem 1. [10 points.]

Let

\[ f(z) = \frac{z}{z^2 - 4}. \]

Expand \( f \) into Laurent series around 0 in the two regions \(|z| < 2\) and \(|z| > 2\) respectively.
Problem 2. [10 points; 5, 5.]

Let $U \subset \mathbb{C}$ be a connected open set.

(i) Show that if $h : U \to \mathbb{C}$ is nonconstant and holomorphic, then $\text{Re } h : U \to \mathbb{R}$ is an open map.
(ii) Let $f : U \to \mathbb{C}$ be holomorphic with $f'(z) \neq 0$ for all $z \in U$. Show that

$$\{\text{Re } f(z) \cdot \text{Im } f(z) : z \in U\}$$

is an open subset of $\mathbb{R}$. 

Problem 3. [10 points.]

Suppose \( f : \Delta(0, 1) \to \mathbb{C} \) is holomorphic such that for all \( z \neq 0 \), we have
\[
|f(z)| \leq -\log |z|.
\]
Show that \( f \equiv 0 \).
Problem 4. [10 points.]

Assume that \( f : \overline{\Delta}(0,1) \to \mathbb{C} \) is continuous, and \( f \) is holomorphic in \( \Delta(0,1) \). Show that if \( f(z) = 0 \) for all \( z = e^{it} \) with \( 0 \leq t < \pi \) then \( f \equiv 0 \).

*Hint: You may wish to work with a convenient auxiliary function.*