HW3 - SOLUTIONS

Q1. Let $\gamma(t) = e^{it}$ for $t \in [-\pi/2, \pi/2]$. The given function $f(z) = ze^{iz}$ is holomorphic on the entire plane and admits a primitive $F(z) = (e^{iz} - ize^{iz})$, hence

$$
\int_{\gamma} ze^{iz} \, dz = F(e^{i\pi/2}) - F(e^{-i\pi/2}) = F(i) - F(-i) = 2e^{-1}.
$$

Q2. The unit half circle is centered at 1. We write $z - 1 = e^{it}$ with $t$ going from $\pi/2$ to $-\pi/2$. By the definition in class we have

$$
\sqrt{z-1} = \exp \left( \frac{1}{2} \log(z-1) \right)
$$

where $\log$ is the principal branch of the logarithm. We therefore have

$$
\sqrt{z-1} = \exp(it/2), \prod_{\gamma} = ie^{it} \prod_{\gamma}
$$

and the integral becomes

$$
\int_{C} \sqrt{z-1} \, dz = \int_{\pi/2}^{-\pi/2} e^{it/2} \cdot ie^{it} \, dt = -i \int_{-\pi/2}^{\pi/2} e^{3it/2} \, dt = -\frac{2}{3} (e^{3\pi i/4} - e^{-3\pi i/4})
$$

$$
= \frac{4i}{3} \cdot \frac{3\pi}{4} = -\frac{2\sqrt{2}i}{3}.
$$

Q3. If $h = fg$ then by direct computation we see that

$$
\frac{h'}{h} = \frac{f'g + fg'}{fg} = \frac{f'}{f} + \frac{g'}{g}.
$$

Applying this repeatedly to

$$
f(z) = c \prod_{\ell=1}^{k} (z - a_\ell)^{m_\ell}
$$

we find

$$
\frac{f'}{f} = \sum_{\ell=1}^{k} \frac{m_\ell}{z - a_\ell}.
$$

Therefore

$$
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum_{k=1}^{\ell} \frac{1}{2\pi i} m_\ell \cdot \int_{\gamma} \frac{dz}{z - a_\ell} = \sum_{\ell=1}^{k} m_\ell \cdot n(\gamma, a_\ell).
$$

If $f$ is a polynomial with roots $a_1, \ldots, a_\ell$ and $R > \max(|a_\ell|)$ we see that letting $\gamma$ be the circle $|z| = R$ we have $n(\gamma, a_\ell) = 1$. Thus

$$
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum_{\ell=1}^{k} m_\ell = \deg f.
$$
Q4.

(i) The integrand has only singularity at 1 inside $|z| = 2$ and the residue at 1 is $\frac{e^4}{2}$. Hence, the integration is

$$2\pi i \left( \frac{e^4}{2} \right) = \pi i e^2.$$

(ii) The integrand has only singularity at $-i$ inside $|z| = 2$ and the residue at $-i$ is $-\sin i$. Hence, the integration is $-2\pi i \sin i = \pi (e - e^{-1})$.

(iii) The integrand is holomorphic inside $|z| = 2$ and hence the integration is 0.

(iv) Note that $|z^5 - iz - 4| \geq 4 - |z|^5 - |z| \geq 2$ on $|z| = 1$. Hence, the integrand is holomorphic inside $|z| = 1$ and the integration is 0.

Q5. Write $f = u + iv$ and $dz = dx + idy$. Thus

$$fdz = (u + iv)(dx + idy) = (u + iv) dx + (-v + iu) dy = P dx + Q dy$$

where

$$P = u + iv, \quad Q = -v + iu.$$

We note that

$$Q_x - P_y = -v_x + iu_x - u_y - iv_y = 0$$

using the Cauchy-Riemann equations. Using Green’s theorem which applies since $P, Q$ are of class $C^1$, we find

$$\int_{\gamma} f dz = \int_{\gamma} P dx + Q dy = \int_{D} (Q_x - P_y) dx dy = 0.$$

Q6. Along the circle $\gamma = \{ w : |w| = R \}$, we have the following identity of differential forms:

$$\frac{dw}{w} + \overline{\frac{dw}{w}} = \frac{(x - iy)(dx + idy) + (x + iy)(dx - idy)}{R^2} = \frac{2(x dx + y dy)}{R^2} = \frac{d(x^2 + y^2)}{R^2} = 0$$

The last equality holds since $x^2 + y^2$ is a constant function over the curve $\gamma$. Let

$$h(w) = \frac{\overline{zf(w)}}{R^2 - \overline{w}},$$

and note that it is holomorphic over $\Delta(0, R + \epsilon)$ where $\epsilon$ is small enough to satisfy $\Delta(0, R + \epsilon) \subset U$ and $(R + \epsilon)(R - |z|) < R$. Thus

$$\int_{\gamma} h(w) dw = 0.$$

Take the conjugate to reduce the problem to showing

$$f(0) = \frac{-1}{2\pi i} \int_{|w|=R} \frac{f(w)}{2 \sqrt{w - \overline{w}}} \overline{dw}$$

(0.1)
where \( d\overline{w} = dx - idy \). Now we express right hand side as
\[
\frac{-1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - \overline{z})} d\overline{w} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)\overline{w}}{(w - \overline{z})w} dw
\]
\[
= \frac{1}{2\pi i} \int_{\gamma} \left( \frac{f(w)\overline{w}}{(w - \overline{z})w} - h(w) \right) dw
\]
\[
= \frac{1}{2\pi i} \int_{\gamma} f(w) w dw
\]
\[
= f(0).
\]
Cauchy’s integral formula was used in the last step.

Q7. Let \( G \) be an open set and \( \gamma \) be a \( C^1 \) loop in \( G \). Suppose \( \phi : \{\gamma\} \times G \rightarrow \mathbb{C} \) is a continuous function and define \( g : G \rightarrow \mathbb{C} \) by
\[
g(z) = \int_{\gamma} \phi(w, z) dw.
\]

\[\begin{itemize}
\item \( g \) is a continuous function : Let \( \ell = L(\gamma) \) be the length of the loop \( \gamma \). For any \( \epsilon > 0 \), we can find \( \delta > 0 \) such that for any \( |h| < \delta \),
\[
|\phi(w, z + h) - \phi(w, z)| < \frac{\epsilon}{\ell},
\]
for all \( w \in \gamma \). Here we are using compactness of \( \text{Im} \gamma \) and the continuity of \( \phi \).
\end{itemize}\]

(Indeed, assuming otherwise. Then, there would exist \( \epsilon > 0 \) such that for all \( \delta \), say \( \delta = \frac{1}{n} \), there exists \( h = h_n \) with \( |h_n| < \frac{1}{n} \) and \( w_n \in \gamma \) such that
\[
|\phi(w_n, z + h_n) - \phi(w_n, z)| \geq \frac{\epsilon}{\ell}.
\]
By compactness of \( \text{Im} \gamma \), we may assume \( w_n \rightarrow w \) after passing to a subsequence. Making \( n \rightarrow \infty \) in the above inequality we obtain \( 0 \geq \frac{\epsilon}{\ell} \) a contradiction.)

Therefore for any \( |h| < \delta \),
\[
|g(z + h) - g(z)| = \left| \int_{\gamma} \left( \phi(w, z + h) - \phi(w, z) \right) dw \right|
\]
\[
< \frac{\epsilon}{\ell} \cdot L(\gamma) = \epsilon.
\]
Assume \( \frac{\partial \phi}{\partial z} \) exists for each \( (w, z) \in \{\gamma\} \times G \) and is continuous. We show

\[\begin{itemize}
\item \( g \) is holomorphic and \( g'(z) = f(z) \) where
\[
f(z) = \int_{\gamma} \frac{\partial \phi}{\partial z}(w, z) dw.
\end{itemize}\]

It is enough to show that
\[
\lim_{h \rightarrow 0} \left( \frac{g(z + h) - g(z)}{h} - f(z) \right) = 0.
\]
Let us denote \( \phi_2 = \frac{\partial \phi}{\partial z}(w, z) \). Given \( \epsilon > 0 \), there exist \( \delta > 0 \) such that for \( |h| < \delta \) we have

\[
|\phi_2(w, z + th) - \phi_2(w, z)| < \frac{\epsilon}{L}
\]

for all \( w \in \gamma \). This follows by the same reasoning as above applied to the function \( \phi_2 \).

Note that

\[
\frac{d}{dt} \phi(w, z + th) = \phi_2(w, z + th)
\]

and hence by the fundamental theorem of calculus, we find

\[
\left| \frac{\phi(w, z + h) - \phi(w, z)}{h} - \phi_2(w, z) \right| = \left| \int_0^1 \phi_2(w, z + th) - \phi_2(w, z) dt \right| \\
\leq \int_0^1 |\phi_2(w, z + th) - \phi_2(w, z)| dt < \frac{\epsilon}{L}
\]

Therefore, for \( |h| < \delta \),

\[
\left| \frac{g(z + h) - g(z)}{h} - f(z) \right| = \left| \int_{\gamma} \left( \frac{\phi(w, z + h) - \phi(w, z)}{h} - \phi_2(w, z) \right) dw \right| < L(\gamma) \frac{\epsilon}{L} = \epsilon
\]

using the basic estimate and the preceding inequality at the last step. This completes the argument.