**Math 220, Problem Set 1.**

*You may use the fact that any complex differentiable function admits as many (continuous) derivatives as you need. We will establish this later.*

**1.** Let $U \subset \mathbb{C}$ be a connected open set. Find all complex differentiable functions $f : U \to \mathbb{C}$ such that

$$(\text{Re } f)^2 + i(\text{Im } f)^2$$

is also complex differentiable.

*Hint:* Harmonic functions will play a role.

**2.** For a complex number $\alpha \in \mathbb{C}$ and positive integer $n > 0$, define the Pochhammer symbol (also known as the rising factorial) as

$$(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1).$$

By definition $(\alpha)_0 = 1$.

Let $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q$ be complex numbers which are not negative integers. The generalized hypergeometric series is defined as

$${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$

(i) Find the radius of convergence of the hypergeometric series.
(ii) Write the series ${}_2F_1(1, 1; 1; z)$ and ${}_2F_1(1, 2; 1; z)$ in closed form. (There is no closed form expression for arbitrary values of the parameters $a_i, b_j$.)
(iii) Verify that differentiation preserves hypergeometric series in the sense that

$$\frac{d}{dz}\left({}_2F_1(a, b; c; z)\right) = \frac{ab}{c} \cdot {}_2F_1(a + 1, b + 1; c + 1; z).$$

*Remark:* Traditionally, one considers the case $p = 2$ and $q = 1$, called the ordinary or Gaussian hypergeometric series, and studied by many illustrious mathematicians including Euler, Gauß, Riemann, etc. They appear in several areas of mathematics, including the study of second-order linear ordinary differential equations. In fact, it can be shown by direct differentiation that the hypergeometric series

$$w = {}_2F_1(a_1, a_2; b; z)$$

satisfies the hypergeometric differential equation

$$z(1 - z)\frac{d^2w}{dz^2} + (b - (1 + a_1 + a_2)z)\frac{dw}{dz} - a_1a_2w = 0.$$

**3.** Let

$$f(z) = \begin{cases} \frac{xy^2(x+iy)}{x^2+y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}.$$

Show that the real and imaginary part of $f$ satisfy the Cauchy Riemann equations at $z = 0$, but $f$ is not complex differentiable at $z = 0$. Why doesn't this contradict the results proved in class?

**4.** Let $U \subset \mathbb{C}$ be a connected open set. A logarithm function $\ell : U \to \mathbb{C}$ is a continuous function such that

$$\exp(\ell(z)) = z \text{ for all } z \in U.$$

We emphasize the continuity assumption in the above definition.

(i) Show that if $\ell$ is a complex differentiable function with $\ell'(z) = \frac{1}{z}$, then $\ell$ must be a logarithm function as defined above, possibly up to a constant. You may wish to begin by differentiating $\exp(\ell(z))$.

(ii) Let $\Delta_1(1)$ be the open disc of radius 1 centered at 1. Using (i), show that the power series

$$L(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(z-1)^k$$

defines a logarithm function on $\Delta_1(1)$.

(iii) Let $a \in \mathbb{C} \setminus \{0\}$ and let $b$ be any logarithm of $a$. Using (ii), derive that

$$L\left(\frac{z}{a}\right) + b$$

is a logarithm function over the ball $\Delta_{|a|}(a)$ of radius $|a|$ centered at $a$.

(iv) Show that if $\ell_1$ and $\ell_2$ are two logarithm functions defined over a connected open set $U$ then $\ell_1 = \ell_2 + 2\pi i n$ for some integer $n$.

(v) Using (iii) and (iv), conclude that any logarithm function $\ell$ is automatically complex differentiable. Show that $\ell'(z) = \frac{1}{z}$.

**5.** Let $U \subset \mathbb{C}$ be open and connected. Assume that $f_1, \ldots, f_n : U \to \mathbb{C}$ are complex differentiable functions such that

$$|f_1|^2 + \ldots + |f_n|^2 = 1.$$

Show that $f_1, \ldots, f_n$ are constant.

*Hint:* You may wish to compute $\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}|f|^2$.