Math 220, Problem Set 2.

1. The dilogarithm is defined as

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

The name comes from the analogy with the expansion $-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ which appeared in the previous problem set.

(i) Show that Li_2 is a holomorphic function in $\Delta(0,1)$.
(ii) Show that Li_2 is injective in $\Delta\left(0, \frac{2}{3}\right)$.

Hint: Use that $z^n - w^n = (z-w)(z^{n-1} + \ldots + w^{n-1})$.

2. Give an example of a biholomorphism between the strip $\{z : -\pi < \mathrm{Im}\ z < \pi\}$ and the slit complex plane $\mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. The answer should be a familiar function.

3. Show that the function $u : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ given by

$$u(z) = \log |z|$$

is harmonic, but it is not the real part of a holomorphic function in $\mathbb{C} \setminus \{0\}$.

Hint: Assume $\mathrm{Re} f = u$, f holomorphic. Consider the function $g = f - \ell$, where ℓ is a branch of the logarithm over a suitable open subset of $\mathbb{C} \setminus \{0\}$. What is the real part of g? What can you conclude about g?

Remark: The above arguments allow you to also solve the first half of the question without any calculation of partial derivatives. (You are free to compute these derivatives if you wish, of course.)

4. (*Qualifying Exam, Spring 2020.*) For $a \in (-1,1)$, let $D_a = \{z \colon |z| < 1,\ \mathrm{Im}\ z > a\}$. For each such a, either find a Möbius transformation of D_a onto the first quadrant Q, or show that such a transformation cannot exist.

5. The arctangent is defined by the power series

$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots.$$

The radius of convergence is $R = 1$ (why?)

(i) Let Log be the principal branch of the logarithm (where is it defined?) Show that

$$g(z) = \frac{1}{2i} \mathrm{Log} \frac{1+iz}{1-iz}$$

is well-defined in $\Delta(0,1)$.
(ii) Show that $g'(z) = \frac{1}{1+z^2}$.

(iii) Conclude that in $\Delta(0,1)$, we have

$$\arctan z = \frac{1}{2i} \operatorname{Log} \frac{1+iz}{1-iz}.$$

6. Let $G = \mathbb{C} \setminus \{x \in \mathbb{R} : |x| \geq 1\}$. Let $f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$.

(i) Show that $f : \mathfrak{h}^+ \to G$ is well-defined.

(ii) Show that $f : \mathfrak{h}^+ \to G$ is injective.

(iii) Show that $f : \mathfrak{h}^+ \to G$ is bijective.

(iv) (*Harder.*) In fact, show that $f : \mathfrak{h}^+ \to G$ is a biholomorphism.

Hint: Solving $f(z) = w$ leads to the expression $z = w + \sqrt{w^2 - 1}$. How do you define the square root? To show the inverse is well-defined, no messy inequalities are needed; a topological argument is quicker.