**Math 220, Problem Set 3.**

**1.** (*Practice with integrals.*)

(i) Consider the path $\gamma(t) = e^{it}$ for $t \in [-\pi/2, \pi/2]$. Compute

$$\int_\gamma ze^{iz}\, dz.$$

(ii) Let $C$ be the half unit circle joining $1+i$ to $1-i$ clockwise. Calculate the integral

$$\int_C \sqrt{z-1}\, dz,$$

where the principal branch of the square root is used for the integrand.

**2.** (*Appreciating Cauchy's formula.*) Calculate the following integrals using Cauchy's integral formula:

(i) $\int_{|z|=2} \frac{e^z}{(z-1)(z-3)^2} dz$
(ii) $\int_{|z|=2} \frac{\sin z}{z+i}\, dz$
(iii) $\int_{|z|=1} \frac{e^z}{(z-2)^3} dz$
(iv) $\int_{|z|=1} \frac{dz}{z^5+iz-4}$.

**3.** (*Quicker proof of Goursat under stronger assumptions.*) Let $\gamma$ be a piecewise $\mathcal{C}^1$-closed curve in a disc $\Delta$ with no self intersections, so that $\gamma$ bounds a domain $D$ with $\overline{D} \subset \Delta$. Let $f : \Delta \to \mathbb{C}$ be a holomorphic function with continuous derivative. Use Green's theorem from multivariable calculus to prove that

$$\int_\gamma f dz = 0.$$

*Remark:* In particular, this recovers the conclusion of Goursat's lemma and also of its Corollary, but under the assumption that $f$ is holomorphic with continuous derivative. (We did not make this assumption in class, so the proof of Goursat we gave had to be a bit more difficult. Nonetheless, as we will see shortly, all holomorphic functions have continuous derivatives, but Goursat's lemma is used in order to arrive at this result.)

*Hint:* Write $f dz = (u+iv)(dx+idy) = Pdx + Qdy$, for suitable $P$ and $Q$, and apply Green's theorem to the $\mathcal{C}^1$-form $Pdx + Qdy$.

**4.** (*Lipschitz estimates for bounded holomorphic functions.*) Let $f : \Delta(0,2) \to \mathbb{C}$ be a holomorphic function such that $|f(z)| \leq M$ for $|z| = 1$. Show that for $w_1, w_2 \in \overline{\Delta}(0, \frac{1}{2})$,

$$|f(w_1) - f(w_2)| \leq 4M|w_1 - w_2|.$$

**5.** (*Differentiation under integral sign. We will use this several times.*) Assume $G \subset \mathbb{C}$ is an open connected set, $\gamma$ is a $\mathcal{C}^1$-curve in $\mathbb{C}$, and $\varphi : \{\gamma\} \times G \to \mathbb{C}$ is continuous. Let

$$g : G \to \mathbb{C}, \quad g(z) = \int_\gamma \varphi(w,z)\, dw.$$

(i) Show that $g$ is continuous. This is first half of Exercise IV.2.2 in Conway.

(ii) If $\frac{\partial \varphi}{\partial z}$ exists at all points $(w, z) \in \{\gamma\} \times G$ and is continuous, then $g$ is holomorphic and

$$g'(z) = \int_\gamma \frac{\partial \varphi}{\partial z}(w, z)\, dw.$$

This is second half of Exercise IV.2.2 in Conway.

(iii) Assume that $f$ is a continuous function defined on $\{\gamma\}$. Show that

$$h(z) = \int_\gamma \frac{f(w)}{w - z}\, dw$$

is holomorphic on $\mathbb{C} \setminus \{\gamma\}$. In fact, show $h$ admits derivatives of all orders on $\mathbb{C} \setminus \{\gamma\}$, and that

$$h^{(k)}(z) = k! \int_\gamma \frac{f(w)}{(w - z)^{k+1}}\, dw.$$

This is Exercise IV.2.3 in Conway.

**6.** (*Continuation of previous question.*)

(iv) Let $f : G \to \mathbb{C}$ be holomorphic. Using (iii) and the local Cauchy integral formula, show that $f$ admits (continuous) derivatives of all orders in $G$.

(v) In particular, if $f : G \to \mathbb{C}$ is a continuous function that admits a holomorphic primitive, then $f$ must be holomorphic.

(vi) (*Morera's theorem.*) Assume that $f : G \to \mathbb{C}$ is continuous and $\int_\gamma f\, dz = 0$ for all piecewise $\mathcal{C}^1$-loops $\gamma$ in $G$. Show that $f$ is holomorphic in $G$.

**7.**

(i) Let $f : G \to \mathbb{C}$ be continuous, and let $R = [a, b] \times [c, d] \subset G$ be a rectangle. Let $R_n = [a + \frac{1}{n}, b - \frac{1}{n}] \times [c + \frac{1}{n}, d - \frac{1}{n}]$ be a rectangle contained in $R$ for $n$ large enough. Show that

$$\int_{\partial R_n} f(z)\, dz \to \int_{\partial R} f(z)\, dz.$$

(ii) (*Qualifying Exam, Fall 2023.*) Let $f : \mathbb{C} \to \mathbb{C}$ be continuous, such that $f$ is holomorphic in $\mathbb{C} \smallsetminus \mathbb{R}$. Show that $f$ is holomorphic in $\mathbb{C}$.

*Hint:* You may wish to show $\int_{\partial R} f\, dz = 0$ for all rectangles $R \subset \mathbb{C}$.