Math 220, Problem Set 4.

All paths/loops are piecewise of class $\mathcal{C}^1$.

1. Let $f : U \to \mathbb{C}$ be a holomorphic function in a connected open set U.

(i) Assume that $|f(z) - 1| < 1$ for all $z \in U$. Show that

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$$

for all closed loops γ in U.

(ii) Assume that U is simply connected and $f(z) \neq 0$ for all $z \in U$. Show that

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$$

for all closed loops γ in U.

(iii) Is it true in general that if $f(z) \neq 0$ for all $z \in U$ then

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$$

for all closed loops γ in U?

2. (*Roots in simply connected regions.*) Assume $f : U \to \mathbb{C}$ is a holomorphic function on a simply connected open set U such that $f(z) \neq 0$ for all $z \in U$. Let $n \geq 2$ be an integer. Show that there is a holomorphic function $g : U \to \mathbb{C}$ such that

$$g(z)^n = f(z).$$

Hint: This has something to do with problem 1(ii).

3. (*Generalized Liouville.*) Assume f is an entire function and p is a polynomial such that

$$|f(z)| \leq |p(z)|$$

for $|z|$ sufficiently large. Using Cauchy's estimate show that f is a polynomial as well.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.

(i) Show that if Re f is bounded (from above or below) then f is constant. You may wish to consider the function $g = e^{\pm f}$.

(ii) Show that if Re $f \leq$ Im f then f is constant. You may wish to reduce this to part (i).

5. (*Qualifying Exam, Spring 2017, small modification.*) Find all entire functions $f : \mathbb{C} \to \mathbb{C}$ such that for all $z \in \mathbb{C}$ we have

$$|f(z)|^2 \leq (\log(1 + |z|))^3 .$$

6. (*Qualifying Exam, Fall 2020.*) Let $f : \mathbb{C} \to \mathbb{C}$ be entire. Show that $\sum_{n=1}^{\infty} \frac{f^{(n)}(z)}{n!}$ converges uniformly on compact subsets of $\mathbb{C}$.

7. (*Towards elliptic functions.*) Show that a function $f : \mathbb{C} \to \mathbb{C}$ which is entire and doubly periodic must be constant. A function f is doubly periodic provided

$$f(z) = f(z + \omega_1) = f(z + \omega_2)$$

for complex numbers ω_1, ω_2 such that $\omega_1/\omega_2 \notin \mathbb{R}$.

Hint: You may wish to consider the values of f on the compact set $P = \{t\omega_1 + s\omega_2, 0 \leq t, s \leq 1\}$. You may also wish to draw a picture.

Remark: Elliptic functions are *meromorphic* functions on $\mathbb{C}$ which are doubly periodic. We will define meromorphic functions later.

8. (*Bernoulli numbers.*) Consider the power series expansion

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \cdot \frac{z^k}{k!}.$$

The expansion holds for $|z| < 2\pi$. The coefficients B_k are called the Bernoulli numbers and they often appear in many areas of mathematics.

(i) Find the first three non-zero Bernoulli numbers.

(ii) Prove that $B_{2k+1} = 0$ for all $k \geq 0$.

(iii) Show that for $p \geq 0$, we have

$$1^p + 2^p + \ldots + N^p = \frac{1}{p+1} \sum_{j=0}^{p} (-1)^j B_j \binom{p+1}{j} \cdot N^{p+1-j}.$$

What does this formula give for $p = 1, 2, 3$?

Hint: For each fixed N, you may wish to examine/compute the series

$$\sum_{p=0}^{\infty} (1^p + \ldots + N^p) \cdot \frac{z^p}{p!}.$$