Math 220, Problem Set 5.

1. (*Qualifying Exam, Spring 2023.*) Let $P_1, \ldots, P_m$ be points on the unit circle. Show that there is a point Q on the unit circle such that

$$P_1Q \cdot P_2Q \cdot \ldots \cdot P_mQ \geq 1.$$

2. (*Qualifying Exam, Spring 2021.*) Let $f : \Delta(0, 2) \to \mathbb{C}$ be a holomorphic function such that $|f(z)| < 1$ for all $z \in \Delta(0, 2)$. Assume

$$f(1) = f(-1) = f(i) = f(-i) = 0.$$

Show that $|f(0)| \leq \frac{1}{15}$.

Hint: MMP for the function $g(z) = f(z)/(z^4 - 1)$ in a disc $\overline{\Delta}(0, r)$, with r close to 2.

3. Let $f : \mathbb{C} \setminus S \to \mathbb{C}$ be a bounded holomorphic function, defined away from a finite set S. Show that f is constant.

4. Show that there is no meromorphic function f on the unit disc $\Delta(0, 1)$ such that f' has a pole of order exactly one at $z = 0$.

5. (*Qualifying Exam, Fall 2023.*) Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$. The proof is very similar, but not entirely identical, to the proof of Casorati-Weierstraß.

6. (*Poles at infinity.*) This question is Conway, Problem 13(a)(b), Chapter V.1.

Let $R > 0$ and $G = \{z : |z| > R\}$. A function $f : G \to \mathbb{C}$ has a removable singularity, a pole, or an essential singularity at infinity if $f(1/z)$ has, respectively, a removable singularity, a pole, or an essential singularity at $z = 0$.

If f has a pole at ∞ then the order of the pole is the order of the pole of $f(1/z)$ at $z = 0$.

(a) Prove that an entire function has a removable singularity at infinity iff it is a constant.
(b) Prove that an entire function has a pole at infinity of order m iff it is a polynomial of degree m.

7. Assume that $f : \mathbb{C} \to \mathbb{C}$ is entire and injective. Show that $f(z) = az + b$. You can solve this problem using the notions introduced in Problem 6 above.

8. Let $f : U \setminus \{a\} \to \mathbb{C}$ be a holomorphic function with an isolated singularity at $a \in U$.

Show that f has a removable singularity, or a pole, or an essential singularity at a respectively if and only if f^2 has a removable singularity, or a pole, or an essential singularity at a.