Math 220, Problem Set 6.

1. Let $f : U \to \mathbb{C}$ be a holomorphic function, let $a \in U$, and assume that $f(a) = 0$, $f'(a) \neq 0$. Show that if r is sufficiently small then

$$\int_{|z-a|=r} \frac{dz}{f(z)} = \frac{2\pi i}{f'(a)}.$$

2. Calculate the integral

$$\int_{|z|=4} \frac{e^z}{(z-1)^2(z-3)^2}\, dz.$$

3. (*Slight generalization of the residue theorem. Conway V.2.5.*) Assume f is holomorphic in U except for simple poles $a_1, \ldots, a_n$ and g is holomorphic in U. Show that

$$\frac{1}{2\pi i} \int_\gamma fg\, dz = \sum_k n(\gamma, a_k)\, g(a_k)\, \mathrm{Res}(f, a_k)$$

for all piecewise C^1 loops γ nullhomotopic in U and avoiding a_i. The residue theorem is recovered setting $g = 1$.

4. (*Qualifying Exam, Spring 2023.*) Calculate the following integral using the residue theorem:

$$\int_0^{2\pi} \cos^{2n} t\, dt.$$

You may wish to first write this integral in terms of $z = e^{it}$.

5. (*Values of the zeta function and the Bernoulli numbers.*) Consider the power series expansion

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \cdot \frac{z^k}{k!}.$$

The expansion holds for $|z| < 2\pi$, and the coefficients B_k are called Bernoulli numbers.

Consider the function

$$f(z) = \frac{1}{z^{2k}(e^z - 1)}.$$

Let γ_m denote the boundary of the rectangle with corners

$$\pm(2m+1)\pi \pm (2m+1)\pi i.$$

(i) Using the residue theorem, compute the integral

$$\int_{\gamma_m} f(z)\, dz.$$

The answer should involve the Bernoulli numbers.

(ii) Using a suitable estimate of the function f along each of the sides of γ_m, show that

$$\lim_{m\to\infty} \int_{\gamma_m} f(z)\,dz = 0.$$

(iii) Use (i) and (ii) to confirm the following formula for the values of the so-called Riemann zeta function in terms of the Bernoulli numbers:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(2\pi)^{2k}(-1)^{k+1}B_{2k}}{2(2k)!}.$$

Remark: The Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $s > 1$ (for now s is real). We showed in (iii) that

$$\zeta(2k) = \frac{(2\pi)^{2k}(-1)^{k+1}B_{2k}}{2(2k)!}.$$

In particular, this generalizes the well-known identity (which is proved in Math 140B using Fourier analysis and Parseval's theorem):

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

6. (*Conway V.2.8, slightly modified. Euler's expression for the cotangent. This will be needed in Math 220B.*) Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Let γ_n be the boundary of the rectangle with corners $n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n - \frac{1}{2} - ni, n + \frac{1}{2} - ni$.

(i) Evaluate

$$\int_{\gamma_n} \frac{\pi \cot \pi z}{z^2 - a^2}\,dz$$

via the residue theorem.

(ii) Show that $|\cot(\pi z)| \leq 3$ on the boundary γ_n. You may wish to consider each side individually.

(iii) Show that

$$\lim_{n\to\infty} \int_{\gamma_n} \frac{\pi \cot \pi z}{z^2 - a^2}\,dz = 0.$$

(iv) Conclude that

$$\pi \cot \pi a = \frac{1}{a} + 2a \sum_{n=1}^{\infty} \frac{1}{a^2 - n^2}.$$

This result is a theorem of Euler from 1748, see *Introductio in analysin infinitorum.*

(v) Setting $a = i$, show the curious identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} = -\frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}}.$$

7. (*Conway V.2.6. Further identities. This will be needed in Math 220B.*) Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Let γ_n be the boundary of the rectangle with corners

$$\pm\left(n + \frac{1}{2}\right) \pm ni.$$

(i) Evaluate

$$\int_{\gamma_n} \frac{\pi \cot \pi z}{(z+a)^2}\, dz$$

via the residue theorem.

(ii) Use (i) to show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = \frac{\pi^2}{\sin^2(\pi a)}.$$

(iii) What does this become for $a = 1/2$? Of course, you can get other fun formulas for different values of a.