Math 220, Problem Set 7.

This is a longer problem set, so please plan accordingly.

Problems 1-3 are similar to the examples in class (and to Conway V.2.7, V.2.10, V.2.12). For the three integrals below, please explain the necessary estimates.

1. (*Applications to real analysis.*) *Using the residue theorem*, compute

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx.$$

Hint: It may be easier to consider the function $f(z) = \frac{1-e^{2iz}}{z^2}$.

2. (*Applications to real analysis.*) *Using the residue theorem*, compute

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx.$$

3. (*Mellin transforms.*) *Using the residue theorem*, compute

$$\int_0^\infty \frac{x^\alpha}{1+x^n}\,dx \text{ where } n > 1+\alpha > 0, n \geq 2 \text{ integer}, \alpha \in \mathbb{R}.$$

4. (*Sum of values of rational functions at the integers.*) Let $R(z) = \frac{P(z)}{Q(z)}$ be a rational function such that $\deg P + 2 \leq \deg Q$. Assume that Q has simple zeros at $a_1, \ldots, a_q$, where $a_j \in \mathbb{C} \setminus \mathbb{Z}$.

Show that

$$\sum_{m=-\infty}^{\infty} R(m) = -\pi \sum_{j=1}^{q} \frac{P(a_j)}{Q'(a_j)} \cdot \cot \pi a_j.$$

Remark: This is a generalization of Problem 6 in the previous problem set, which corresponds exactly to the case $R(z) = \frac{1}{a^2 - z^2}$, as one can easily check. The proof is also very similar. The point of the question is to show how far these techniques can be pushed.

(i) Let γ_n be the square with corners

$$\pm\left(n+\frac{1}{2}\right) \pm i\left(n+\frac{1}{2}\right).$$

Show that there exist constants $M_1, M_2 > 0$ such that if n is sufficiently large, and z is on the curve γ_n, we have

$$|\pi \cot \pi z| \leq M_1$$

and

$$|R(z)| \leq \frac{M_2}{|z|^2}.$$

In fact, you should have established the first inequality in the previous problem set, so only the second inequality needs a brief justification.

(ii) Show that

$$\lim_{n\to\infty}\int_{\gamma_n} R(z)\pi\cot\pi z\,dz = 0.$$

(iii) Show that $\pi\cot\pi z$ has poles at all integers $m\in\mathbb{Z}$ with residue equal to 1.

(iv) Find the poles and residues of $R(z)\pi\cot\pi z$ at all the integers $m\in\mathbb{Z}$, and also at the points $a_1,\ldots,a_q$.

(v) Conclude the argument.

(vi) As an application, write down the value of the sum

$$\sum_{m=-\infty}^{\infty}\frac{1}{m^2+m+1}.$$

You do not need to simplify the answer.

5. (*Residues at infinity.*) Assume that f has finitely many isolated singularities in $\mathbb{C}$.

(i) Show that for all $R>0$,

$$\int_{|z|=R} f dz = -2\pi i\sum_j \text{Res }(f(z)dz, a_j)$$

where a_j's are the singularities of f outside the circle $|z|=R$, including ∞. (We assume there are no singularities when $|z|=R$. The circle is positively oriented.)

(ii) Find the residues of

$$f(z)\,dz = (z-a)^k\,dz$$

over $\widehat{\mathbb{C}}$, for k any integer and $a\in\mathbb{C}$.

(iii) Using (i), compute

$$\int_{|z|=5}\frac{z^3\,dz}{(z-1)(z-2)(z-3)(z-4)}.$$

The moral is that considering residues at ∞ allows for faster computations.

6. (*Meromorphic functions on the Riemann sphere.*) Show that a function f which is meromorphic over the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ must be a rational function.

By definition, a meromorphic function over $\widehat{\mathbb{C}}$ has isolated poles which could occur at certain points in $\mathbb{C}$ and possibly also at ∞. (Poles at ∞ were defined in a previous problem set.)

Hint: First note that f must have finitely many poles, by compactness of $\widehat{\mathbb{C}}$. Construct a polynomial g with zeroes exactly at the poles of f in $\widehat{\mathbb{C}}\setminus\{\infty\}$. Show that the product fg is a polynomial.

Remark: This exercise illustrates (early) similarities between *complex analysis* (which studies meromorphic functions, among others) and *algebraic geometry* (which studies rational functions, among others).