Math 220, Problem Set 8.

1. (*A particular case of the argument principle that can be proved directly. Logarithmic derivatives.*)

(i) If $h = fg$, show that
$$\frac{h'}{h} = \frac{f'}{f} + \frac{g'}{g}$$
whenever f, g are holomorphic and non-zero.

In general, for a holomorphic function f, the meromorphic function $\frac{f'}{f}$ is called *the logarithmic derivative*. Thus, taking logarithmic derivatives turns products into sums.

(ii) Assume that
$$f(z) = c\prod_{\ell=1}^{k}(z - a_\ell)^{m_\ell}$$
is a polynomial with roots at $a_1, \ldots, a_k$ with multiplicities $m_1, m_2, \ldots, m_k$. Show that
$$\frac{f'(z)}{f(z)} = \sum_{\ell=1}^{k} \frac{m_\ell}{z - a_\ell}.$$

(iii) Derive that for any piecewise C^1 loop γ avoiding $a_1, \ldots, a_k$ we have
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{\ell=1}^{k} m_\ell \cdot n(\gamma, a_\ell).$$

In particular, if R is sufficiently large, and $\gamma(t) = Re^{it}$ for $0 \le t \le 2\pi$, show that
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \deg f = \#\text{zeros of } f \text{ counted with multiplicity.}$$

2. (*Rouché's theorem.*) Find the number of zeros of the polynomial $z^4 + 5z + 3$ inside the annulus $1 < |z| < 2$.

3. (*Qualifying Exam, Spring 2021.*) How many solutions does the equation
$$z^3 \sin z + 5z^2 + 2 = 0$$
have inside the unit disc $|z| < 1$?

4. (*Qualifying Exam, Fall 2020.*) Let $\lambda > 1$ be a real number.

(i) Show that all solutions to the equation
$$z + e^{-z} = \lambda$$
in the right half plane $\mathrm{Re}\, z > 0$ must be contained in the disc $|z - \lambda| < 1$.

(ii) Show that $z + e^{-z} = \lambda$ has exactly one solution in the right half plane.

(iii) Deduce that the solution in (ii) must be real.

5. (*Rouché's theorem.*) Find the number of zeroes of $z^4 + 3z^2 + z + 1$ inside the unit disc.

Hint: The dominant term is not a monomial.

6. (*An application of complex analysis to algebra.*) Using Rouché's theorem, derive Perron's criterion: a polynomial

$$f(x) = x^n + a_1 x^{n-1} + \ldots + a_n \in \mathbb{Z}[x]$$

with

$$|a_1| > 1 + |a_2| + \ldots + |a_n|, \ a_n \neq 0$$

is necessarily irreducible over $\mathbb{Z}[x]$.

Hint: Use Rouché to determine how many roots of f are inside the unit disc and how many are outside.