Math 220A - Fall 2023 - Final

Name: ____________________

Student ID: ____________________

Instructions:

Please print your name and student ID.

You may use any results proved in class or from Math 140 without proof.

There are 7 questions which are worth 70 points. You have 180 minutes to complete the test.

Question	Score	Maximum
1		8
2		10
3		9
4		13
5		8
6		10
7		12
Total		70

Problem 1. [*8 points.*]

Consider the function $f(z) = ze^{3-2z} - 1$. Show that f has exactly one zero inside the disc $\Delta(0, 1)$.

Problem 2. [*10 points.*]

Let $f : \mathbb{C} \to \mathbb{C}$ be entire. Show that the series

$$g(z) = \sum_{n=0}^{\infty} \left(\frac{f^{(n)}(z)}{n!} \right)^2$$

defines an entire function, where $f^{(n)}$ denotes the n^{th} derivative of f.

Problem 3. [*9 points.*]

Consider

$$f(z) = z^n + a_1 z^{n-1} + \ldots + a_n.$$

Show that there exists c with $|c| = 1$ such that

$$|f(c)| \geq 1.$$

Problem 4. [*13 points.*]

Let $f(z) = \frac{P(z)}{Q(z)}$ be a rational function with $\deg P \leq \deg Q - 2$ such that Q has no zeros along the non-negative real axis. Show that

$$\int_0^\infty f(x)\,dx = -\sum_{a \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}} \mathrm{Res}_{z=a}(f(z)\log z),$$

where for $z \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ we set $\log z = \log r + i\theta$ and $\theta \in (0, 2\pi)$.

You may wish to integrate along the "keyhole" contour. Make sure you present the necessary estimates.

(Extra page.)

Problem 5. [*8 points.*]

Let $f : U \setminus \{a\} \to \mathbb{C}$ be a holomorphic function in a punctured neighborhood of a, having $z = a$ as an essential singularity. Show that f is not injective.

Problem 6. [*10 points; 5, 5.*]

Let f be a continuous function in the closed unit disc $\overline{\Delta}$, holomorphic inside the unit disc Δ. Assume that

$$|f(z)| = 1 \text{ for all } |z| = 1.$$

(i) If f is not constant, show that f must have a zero inside Δ.

(ii) Possibly using (i) to a suitable function, show that if f has only one zero at 0, of order 1, then $f(z) = \alpha z$ for some $\alpha \in \mathbb{C}$.

Problem 7. [*12 points; 1, 4, 4, 3.*]

Assume that f is entire and $f(z) = f(z+1)$ such that $|f(z)| \leq e^{|z|}$. Show that f is constant.

(i) Consider

$$g(z) = \frac{f(z) - f(0)}{\sin \pi z}.$$

Check that $g(z+1) = -g(z)$.

(ii) Show that g can be extended to an entire function.

(iii) Using suitable estimates, show that

$$\lim_{y\to\infty} g(x+iy) = \lim_{y\to-\infty} g(x+iy) = 0, \qquad \text{for all } 0 \le x \le 1.$$

Also show that g is bounded in the strip $0 \le \operatorname{Re} z \le 1$.

(iv) Conclude that $g = 0$ hence f is constant.