Math 220 A — Lecture 10

November 1, 2023

Types of singularities

$f : \Delta^*(a, R) \longrightarrow \mathbb{C}$, holomorphic.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad \text{Laurent series.}$$

Terminology

(i) the coefficient of $(z-a)^{-1}$:

$$a_{-1} = \operatorname*{Res}_{z=a} f = \text{residue} = \frac{1}{2\pi i} \int_{\gamma_r} f \, dz$$

(ii) $$\sum_{n=-\infty}^{-1} a_n (z-a)^n = \text{principal part.}$$

Three cases

$\boxed{A}$ $a_k = 0 \quad \forall\, k < 0 \iff$ Taylor expansion

$\iff f$ extends holomorphically across a

Removable singularity

$\boxed{B}$ $a_k = 0 \quad \forall\, k < -N, \quad a_{-N} \neq 0$

Pole of order N.

$\boxed{C}$ $a_k \neq 0$, $k < 0$ happens infinitely often

Essential singularity

Case A a removable singularity

Theorem A $f : \Delta^*(a, R) \longrightarrow \mathbb{C}$ holomorphic. TFAE

$\boxed{i}$ f extends holomorphically across a

$\boxed{ii}$ f extends continuously across a

$\boxed{iii}$ f bounded near a

$\boxed{iv}$ $\lim_{z \to a} f(z) \cdot (z - a) = 0.$

Proof $\boxed{i} \Rightarrow \boxed{ii} \Rightarrow \boxed{iii} \Rightarrow \boxed{iv}$ obvious

$\boxed{iv} \Rightarrow \boxed{i}$. WLOG $a = 0$, else work with $f(z + a)$.

We show $a_k = 0 \ \forall\, k < 0$. Fix $\varepsilon > 0$. Since

$$\lim_{z \to 0} z f(z) = 0 \implies |f(z)| < \frac{\varepsilon}{|z|} \quad \text{if } |z| < \delta.$$

We have for $0 < r < \delta < R$:

$$|a_k| = \left| \frac{1}{2\pi i} \int_{|z| = r} \frac{f(z)}{z^{k+1}} \, dz \right| \leq \frac{1}{2\pi} \cdot \frac{\varepsilon}{r} \cdot \frac{1}{r^{k+1}} \cdot 2\pi r$$

$$= \frac{\varepsilon}{r^{k+1}}.$$

If $k=-1$: $|a_{-1}| < \varepsilon \ \forall \varepsilon > 0 \Rightarrow a_{-1} = 0$.

If $k < -1$, take $\varepsilon = 1$, $|a_k| < \frac{1}{r^{k+1}}$. Make $r \to 0$

to obtain $a_k = 0$. since $k < -1$.

Example $f: \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, $a \in \mathcal{U}$

$$g(z) = \begin{cases} \frac{f(z) - f(a)}{z - a}, & z \neq a \\ f'(a), & z = a \end{cases}$$

is holomorphic.

Indeed $\frac{f(z) - f(a)}{z - a}$ has a removable singularity at a

by item $\boxed{IV}$. & g is the continuous / holomorphic

extension across a.

Remark In hindsight, Goursat$^+$ & Cauchy$^+$ were equivalent to

Goursat & Cauchy.

Case B a pole of order N.

$$f(z) = (z-a)^{-N}\, g(z)$$ holomorphic

$$g(z) = \sum_{k=0}^{\infty} a_{k-N}\,(z-a)^k, \qquad g(a) = a_{-N} \neq 0.$$

Zeros versus Poles

$$\frac{1}{f(z)} = (z-a)^N \cdot \frac{1}{g(z)}, \quad \frac{1}{g} \text{ holomorphic near } a$$

f pole of order N at $a \iff \frac{1}{f}$ zero of order N at a

Lemma B $f : \Delta^*(a,R) \to \mathbb{C}$ holomorphic TFAE

[i] a is a pole

[ii] $\lim_{z\to a} f(z) = \infty$.

Proof [i] $\Rightarrow$ [ii] Write $f(z) = (z-a)^{-N} g(z)$.

Since $g(a) \neq 0 \Rightarrow |g(z)| \geq M > 0$ in $|z-a| < \delta$.

$$\Rightarrow |f(z)| = \frac{|g(z)|}{|z-a|^N} \geq \frac{M}{|z-a|^N}. \quad \text{Make } z \to a \text{ to}$$

conclude $\lim_{z \to a} f(z) = \infty$.

[ii] $\Rightarrow$ [i] Note $\lim_{z \to a} f(z) = \infty \Rightarrow \lim_{z \to a} \frac{1}{f(z)} = 0$

$\Rightarrow \frac{1}{f}$ bounded near a $\Rightarrow \frac{1}{f}$ can be extended across a

holomorphically. Note the extension vanishes at a, say

of order N $\Rightarrow f$ has a pole at a of order N.

Definition $S \subseteq U$ discrete. A function f holomorphic in

$U \setminus S$, with at worst poles at S is called meromorphic.

Example

polynomials

[i] $f(z) = \frac{P(z)}{Q(z)}$, $U = \mathbb{C}$ meromorphic.

[ii] $f(z) = \frac{1}{\sin \frac{1}{z}}$, $U = \mathbb{C}^{\times}$.

Check $z = \frac{1}{n\pi}$, $n \in \mathbb{Z}^{\times}$ are poles. These do not accumulate in $U = \mathbb{C} \setminus \{0\}$. Thus f meromorphic in $U = \mathbb{C}^\times$.

Case C $\quad f : \Delta^*(a, R) \longrightarrow \mathbb{C}$ holomorphic

a essential singularity $\quad$ e.g. $f(z) = e^{\frac{1}{z}}$.

Example $\quad f(z) = e^{\frac{1}{z}} = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{z^k} \cdot \Rightarrow$

$\Rightarrow$ $a = 0$ is essential singularity.

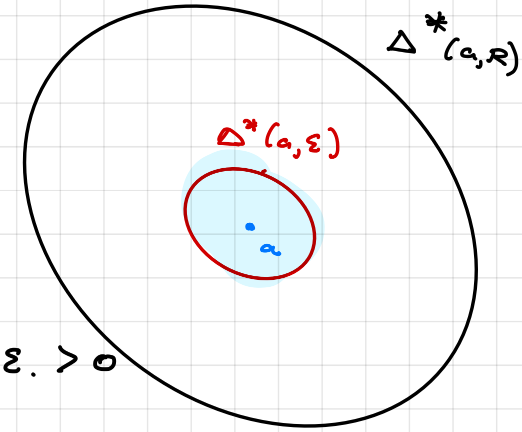
Remark $\quad f$ cannot be bounded or go to ∞.

(see cases A & B)

Question $\quad$ How does f behave near a?

Theorem (Big Picard Theorem) — 220C.

$\forall\ \Delta^*(a,\varepsilon) \subseteq \Delta^*(a,R)$, $f(\Delta^*(a,\varepsilon)) = \mathbb{C}$ or $\mathbb{C}\setminus\{\text{point}\}$.



Example $f(z) = e^{\frac{1}{z}}$. $a = 0$.

Claim $f(\Delta^*(0,\varepsilon)) = \mathbb{C}\setminus\{0\}$. $\forall\ \varepsilon > 0$

Proof $y \neq 0$: $y = e^{\frac{1}{z}}$, $z \in \Delta^*(0,\varepsilon)$.

$\iff \frac{1}{z} = \log y + 2n\pi i$ for any choice of $\log$

$\iff z = \dfrac{1}{\log y + 2n\pi i} \in \Delta^*(0,\varepsilon)$ if $n \gg 0$.

Weaker version:

Theorem (Casorati – Weierstraß) $f: \Delta^*(a,R) \longrightarrow \mathbb{C}$

TFAE $\boxed{i}$ f has essential singularity at a

$\boxed{ii}$ $\forall\ \Delta^*(a,\varepsilon) \subseteq \Delta^*(a,R)$, $f(\Delta^*(a,\varepsilon))$ is dense in $\mathbb{C}$.

Proof $\boxed{\text{I}} \Rightarrow \boxed{\text{II}}$ Assume for some $\varepsilon > 0$, the set

$f(\Delta^*(a,\varepsilon))$ is not dense in $\mathbb{C}$. Then $\exists\, \Delta(\lambda,\rho)$

$$(*).\quad f(\Delta^*(a,\varepsilon)) \cap \Delta(\lambda,\rho) = \phi.$$

Define $g = \dfrac{1}{f-\lambda}$ in $\Delta^*(a,\varepsilon)$. By $(*)$ we know

$$|f-\lambda| \geq \rho \text{ in } \Delta^*(a,\varepsilon) \Rightarrow |g| \leq \frac{1}{\rho} \text{ in } \Delta^*(a,\varepsilon)$$

Thm A
$\Rightarrow$ a is removable singularity for g. But

$$f = \lambda + \frac{1}{g}. \quad (+)$$

If a is not a zero for $g \Rightarrow \dfrac{1}{g}$ holomorphic $\overset{(+)}{\Rightarrow}$
f extends holomorphically across $a \Rightarrow$ removable singularity.

If a is a zero for $g \Rightarrow \dfrac{1}{g}$ has a pole at $a \overset{(+)}{\Rightarrow}$
$\Rightarrow f$ has pole at a.

Both cases are impossible.

[iii] $\Rightarrow$ [i] Assume a removable singularity $\Rightarrow$

Thm A

$\Rightarrow$ f bounded near a $\Rightarrow$ $\exists\, M>0,\ \varepsilon>0$ with

$$|f(z)| < M \quad \text{in} \quad \Delta^*(a,\varepsilon)$$

$\Rightarrow f(\Delta^*(a,\varepsilon))$ cannot be dense.

Assume a pole $\overset{\text{Lemma B}}{\Rightarrow}$ $\lim\limits_{z\to a} f(z) = \infty \Rightarrow$

$\Rightarrow \exists\ \varepsilon>0,\ |f(z)| \geq 1$ in $\Delta^*(a,\varepsilon) \Rightarrow$

$\Rightarrow f(\Delta^*(a,\varepsilon))$ cannot be dense.

Thus a is essential singularity.

Felice Casorati

1835 – 1890

Karl Weierstraß

1815 – 1897

Émile Picard

1856 – 1941