Math 220 A – Lecture 11

November 8, 2023

1. Residues (Conway V. 2)

a singularity for f

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k \qquad \text{Laurent series}$$

$$a_{-1} = \text{Res}(f,a) = \text{residue}$$

Problem: Compute $\text{Res}(f,a)$

Method 0 Explicit Laurent expansion

Example $f(z) = \dfrac{z}{\sin^4 z}$, $\text{Res}(f,0) = ?$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = z\left(1 - \frac{z^2}{6} + \frac{z^4}{120} - \dots\right)$$

$$\sin^4 z = z^4\left(1 - \frac{z^2}{6} + \frac{z^4}{120} - \dots\right)^4$$

$$= z^4\left(1 - \frac{4z^2}{6} + \dots\right)$$

$$f(z) = \frac{z}{z^4\left(1 - \frac{4z^2}{6} + \dots\right)} = \frac{1}{z^3}\cdot\left(1 + \frac{4z^2}{6} + \dots\right)$$

$$= \frac{1}{z^3} + \frac{4}{6}\cdot\frac{1}{z} + \dots$$

$$\Rightarrow \text{Res}(f,0) = \frac{2}{3}.$$

Method 1

$f(z) = \dfrac{g(z)}{h(z)}$, g, h holomorphic

Assume a simple zero for h $\Rightarrow$ a simple pole for f.

$$\mathrm{Res}(f,a) = \lim_{z \to a} (z-a) f(z)$$

$$= \lim_{z \to a} (z-a) \frac{g(z)}{h(z) - h(a)}$$

$$= \lim_{z \to a} \frac{g(z)}{\dfrac{h(z)-h(a)}{z-a}} = \frac{g(a)}{h'(a)}$$

Conclusion: $\boxed{\mathrm{Res}(f,a) = \dfrac{g(a)}{h'(a)}.}$ if $h'(a) \neq 0$.

Example $f(z) = \dfrac{z - \sin z}{z^2 \sin z}$

- poles $z = 0$, $z = n\pi$, $n \neq 0$, $n \in \mathbb{Z}$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \implies \frac{\sin z}{z} \to 1 \text{ as } z \to 0$$

- $z = 0$ is removable since $\implies \dfrac{z - \sin z}{z^3} \to \dfrac{1}{3!}$ as $z \to 0$

$$\lim_{z \to 0} \frac{z - \sin z}{z^2 \sin z} = \lim_{z \to 0} \underbrace{\cancel{\frac{z - \sin z}{z^3}}}_{\frac{1}{6}} \cdot \underbrace{\cancel{\frac{z}{\sin z}}}_{1.} = \frac{1}{6}.$$

Since $z = 0$ is removable $\implies \operatorname{Res}(f, 0) = 0$.

- $z = n\pi$, $n \neq 0$. Take $g(z) = \dfrac{z - \sin z}{z^2}$

$$h(z) = \sin z$$

$$\implies g(n\pi) = \frac{1}{n\pi}, \quad h'(n\pi) = \cos z \Big|_{z = n\pi} = (-1)^n.$$

$$\implies \operatorname{Res}(f, n\pi) = \frac{g(n\pi)}{h'(n\pi)} = \frac{1}{n\pi} \cdot (-1)^n.$$

Method 2

$$f(z) = \frac{g(z)}{(z-a)^k} \quad \Longrightarrow \quad \mathrm{Res}(f,a) = \frac{g^{(k-1)}(a)}{(k-1)!}$$

holomorphic

Write $\displaystyle g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!} (z-a)^n.$

coeff. of $(z-a)^{-1}$ in $f \iff$ coeff. of $(z-a)^{k-1}$ in g.

This equals $\dfrac{g^{(k-1)}(a)}{(k-1)!}$

Example

$$f(z) = \frac{z}{(z^2+1)^2} \quad \Longrightarrow \quad \mathrm{Res}(f,i) = ?$$

$$f(z) = \frac{g(z)}{(z-i)^2}, \quad g(z) = \frac{z}{(z+i)^2}. \quad \Longrightarrow \quad g'(i) = 0 \quad (\text{check})$$

$$\mathrm{Res}(f,i) = g'(i) = 0.$$

2. Residue Theorem (Conway V. 2)

Toy Example $f : \Delta^*(a, R) \longrightarrow \mathbb{C}$, holomorphic.

$$\Rightarrow \boxed{\int_{\gamma_s} f(z)\, dz = 2\pi i \ \operatorname{Res}(f, a)} \quad \text{where } \gamma_s = \partial \Delta(a, s).$$

↳ this also follows from the proof of Laurent expansion (Lecture 10).

Proof Write

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k.$$

This converges uniformly on compact sets, so we can integrate

$$\Rightarrow \int_{\gamma_s} f\, dz = \sum_{k=-\infty}^{\infty} a_k \int_{\gamma_s} (z-a)^k\, dz$$

$k \neq -1$, integral $= 0$

$k = -1$, integral $= 2\pi i$

$$= 2\pi i \ a_{-1} = 2\pi i \ \operatorname{Res}(f, a).$$

$k \neq -1$: $(z-a)^k$ admits a primitive $\dfrac{(z-a)^{k+1}}{k+1}$. $\Rightarrow$ zero integral.

$k = -1$: $\displaystyle\int_{\gamma_s} \frac{dz}{z-a} = 2\pi i \ n(\gamma_s, a) = 2\pi i$

Residue Theorem

$U \subseteq \mathbb{C}$ open connected, S discrete

- $\gamma \overset{U}{\sim} 0$, $\{\gamma\} \subseteq U \setminus S$.
- f holomorphic in $U \setminus S$, singularities at S.

Then

$$\frac{1}{2\pi i} \int_\gamma f \, dz = \sum_{s \in S} \operatorname{Res}(f, s) \cdot n(\gamma, s).$$

↳ Conway $\underline{V}$. 2.2. (S = finite)

Remarks

$\boxed{i}$ $S = \emptyset \Rightarrow \int_\gamma f \, dz = 0 \Rightarrow$ Cauchy's Theorem (Homotopy)

$\boxed{ii}$ $S = \{a\}$, $\gamma = \gamma_r$ = small circle near a $\Rightarrow$

recovers the toy example.

$\boxed{iii}$ $S = \{a\}$, $f(z) = \dfrac{g(z)}{(z-a)^{k+1}}$, g holomorphic, $\gamma \overset{U}{\sim} 0$

$$\frac{1}{2\pi i} \int_\gamma f \, dz = \frac{1}{2\pi i} \int_\gamma \frac{g(z)}{(z-a)^{k+1}} \, dz = \operatorname{Res}(g, a) \cdot n(\gamma, a)$$

$$= \frac{g^{(k)}(a)}{k!} \cdot n(\gamma, a) \text{ by Method 2.}$$

This recovers CIF for derivatives. ← Lecture 8

IV The sum in RHS is finite

Claim $\{ s \in S, \ n(\gamma, s) \neq 0 \}$ finite

Proof $W = \{ z \in \mathbb{C} \setminus \gamma : n(\gamma, z) \neq 0 \}$.

- W = union of components of $\mathbb{C} \setminus \gamma$ = open
- W bounded

Lecture 6

- $W \subseteq \mathcal{U}$. Indeed if $z \in W$, $z \notin \mathcal{U} \Rightarrow n(\gamma, z) \neq 0$. But

$$n(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z} = 0 \text{ by Cauchy}$$

using $\zeta \to \dfrac{1}{\zeta - z}$ holomorphic in $\mathcal{U}$, $\gamma \overset{\mathcal{U}}{\sim} 0$.

$K = W \cup \{\gamma\} \subseteq \mathcal{U}$ closed & bounded

K compact in $\mathcal{U}$, S discrete in $\mathcal{U}$

$\Rightarrow K \cap S$ = finite.

Example

$$\int_{|z|=3} \frac{z+1}{z^2(z-1)}\, dz$$

Take $U = \Delta(0,4)$, $S = \{0,1\}$, $f(z) = \dfrac{z+1}{z^2(z-1)}$.

- $\operatorname{Res}(f,0) = \operatorname*{Res}_{z=0} \dfrac{\frac{z+1}{z-1}}{z^2} = \left(\dfrac{z+1}{z-1}\right)'\Big/_{z=0} = -2$.

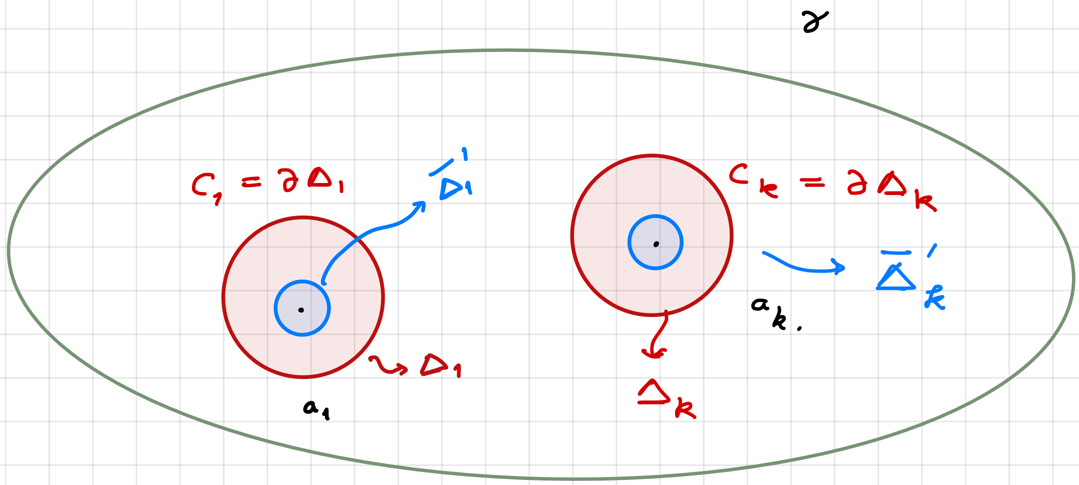
by Method 2 of computing residues

- $\operatorname{Res}(f,1) = \operatorname*{Res}_{z=1} \dfrac{z+1}{z^2(z-1)} = \dfrac{(z+1)/_{z=1}}{(z^2(z-1))'/_{z=1}} = \dfrac{2}{1} = 2$

by Method 1 of computing residues

Thus $\displaystyle\int_{|z|=3} f\, dz = 2\pi i \left(\operatorname{Res}(f,0) + \operatorname{Res}(f,1)\right) = 0.$

Naive "proof" γ simple closed curve. Let $S = \{a_1, \dots, a_k\}$.



Let $C_i = \partial\Delta_i$ be circles centred at a_i, $\Delta_i \subseteq \mathcal{U}$.

Let $\overline{\Delta_i'} \subseteq \Delta_i$. Let $\mathcal{U}' = \mathcal{U} \setminus \bigcup_{i=1}^{k} \overline{\Delta_i'}$ = open.

Let $\eta = \sum_{i=1}^{k} \partial C_i$. Assume we could show

$\gamma \overset{u'}{\sim} \eta$ and $n(\gamma, a_i) = 1$.

Then by Cauchy applied to $f|_{u'}$ we'd have

$$\int_\gamma f\,dz = \int_\eta f\,dz = \sum_{i=1}^{k} \int_{C_i} f\,dz$$

$$= 2\pi i \sum_{i=1}^{k} \operatorname{Res}(f, a_i) \quad \text{(toy example).}$$

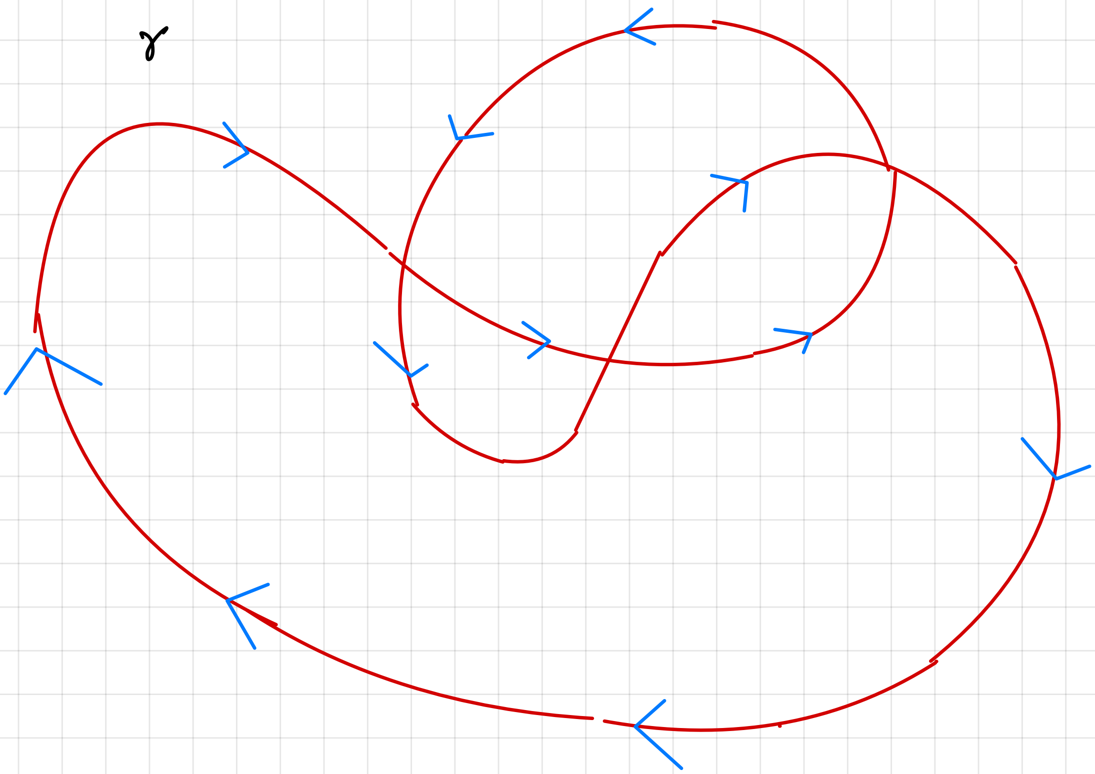
$$= 2\pi i \sum_{i=1}^{k} \operatorname{Res}(f, a_i)\, n(\gamma, a_i).$$

Issues : $\boxed{a}$ η is not a path, but chain

$\boxed{b}$ $\gamma \overset{u'}{\sim} \eta$ and $n(\gamma, a_i) = 1$ need proofs

$\boxed{c}$ how about more complicated curves ?

γ



The proof of the residue theorem requires new ideas.

3. Chains

Terminology $\quad U^* \subseteq \mathbb{C}$, $\gamma^* = \sum_{i=1}^{\ell} m_i \gamma_i$ $\quad C^1$-chain

[a] $$\int_{\gamma^*} f \, dz = \sum_{i=1}^{\ell} m_i \int_{\gamma_i} f \, dz$$

[b] $$n(\gamma^*, a) = \sum_{i=1}^{\ell} m_i \, n(\gamma_i, a)$$

Definition $\quad \gamma^* \overset{U^*}{\approx} 0$ if $n(\gamma^*, a) = 0 \ \forall a \notin U^*$.

(we say γ^* is *nullhomologous* in U^*).

Remark [1] γ^* loop in U^*. Then

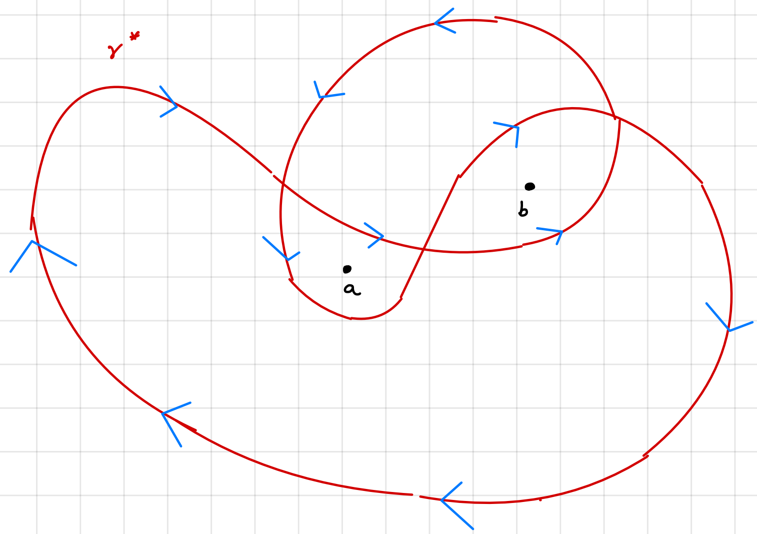
$$\gamma^* \overset{U^*}{\sim} 0 \implies \gamma^* \overset{U^*}{\approx} 0.$$

Indeed if $a \notin U^*$ then

$$n(\gamma^*, a) = \frac{1}{2\pi i} \int_{\gamma^*} \frac{dw}{w-a} = 0$$

by homotopy form of Cauchy applied to $\gamma^* \overset{U^*}{\sim} 0$ and to the holomorphic function $\dfrac{1}{w-a}$ in U^*. ($a \notin U^*$)

[ii] the converse is false $U^* = \mathbb{C} \setminus \{a, b\}$



Check $\gamma^* \overset{U^*}{\approx} 0$. Indeed $n(\gamma^*, a) = n(\gamma^*, b) = 0$.

To see this, find two subloops of γ^* going clockwise & counterclockwise around a. Do the same for b.

However $\gamma^* \overset{U^*}{\not\sim} 0$.

Remark* In algebraic topology, one learns that 1^{st} homology is the abelianization of π_1. (which is defined via homotopy). Thus we expect a connection between $\approx$ and $\sim$.

Enhanced Cauchy's Theorem (Homology Cauchy)

We seek to prove a "homology" version of Cauchy:

Theorem $f: U^* \longrightarrow \mathbb{C}$ holomorphic, $\gamma^* \overset{U^*}{\approx} 0$. Then

$$\int_{\gamma^*} f\, dz = 0.$$

$\hookrightarrow$ Conway IV. 5. 7.

Remark By the above remarks, we see

Homology Cauchy $\Rightarrow$ Homotopy Cauchy

Remark We will see next that

Homology Cauchy $\Rightarrow$ Residue Thm.