Math 220 A — Lecture 12

November 13, 2023

Enhanced Cauchy's Theorem (Homology Cauchy)

We seek to prove a "homology" version of Cauchy:

Theorem $f : \mathcal{U}^* \longrightarrow \mathbb{C}$ holomorphic, $\gamma^* \overset{\mathcal{U}^*}{\approx} 0$. Then

$$\int_{\gamma^*} f \, dz = 0.$$

$\rightsquigarrow$ Conway IV. 5. 7.

Recall $\gamma^* \overset{\mathcal{U}^*}{\approx} 0$ means $n(\gamma^*, a) = 0 \;\; \forall \, a \notin \mathcal{U}^*$.

Remark We will see next that

Homology Cauchy $\Longrightarrow$ Residue Thm.

Residue Theorem $U \subseteq \mathbb{C}$ open connected, S discrete

- $\gamma \overset{U}{\sim} 0$, $\{\gamma\} \subseteq U \setminus S$.
- f holomorphic in $U \setminus S$, singularities at S.

Then

$$\frac{1}{2\pi i} \int_\gamma f \, dz = \sum_{s \in S} \mathrm{Res}(f, s) \cdot n(\gamma, s).$$

$\hookrightarrow$ Conway $\underline{\overline{V}}$. 2.2

Proof of residue theorem

We let f holomorphic in $\mathcal{U} \setminus S$, $\gamma \overset{\mathcal{U}}{\sim} 0$. We want

$$\frac{1}{2\pi i} \int_\gamma f \, dz = \sum_{s \in S} \operatorname{Res}(f, s) \cdot n(\gamma, s).$$

We saw RHS is finite since

$$\{ s \in S : n(\gamma, s) \neq 0 \} \text{ is finite.}$$

Enumerate this set to be $\{a_1, \dots a_k\}$, $m_i = n(\gamma, a_i) \neq 0$.

Let Δ_i be small disjoint discs near a_i. $\Delta_i \subseteq \mathcal{U}$; $\Delta_i \cap S = \{a_i\}$.

Define
- $\mathcal{U}^* = \mathcal{U} \setminus S$
- $\gamma^* = \gamma + \sum_{i=1}^{k} (-m_i) C_i$ where $C_i = \partial \Delta_i$

(positive orientation)

Claim $\gamma^* \overset{\mathcal{U}^*}{\approx} 0$

Homology Cauchy for $(\mathcal{U}^*, \gamma^*)$ $\Rightarrow \int_{\gamma^*} f \, dz = 0$

$$\Rightarrow \frac{1}{2\pi i} \int_\gamma f \, dz = \sum_{i=1}^{k} m_i \cdot \frac{1}{2\pi i} \int_{C_i} f \, dz$$

$$= \sum_{i=1}^{k} m_i \operatorname{Res}(f, a_i) \text{ by toy example}$$

last time. QED.

Proof of the claim Want $n(\gamma^*, a) = 0$ if $a \notin U^*$.

[i] if $a \notin U$. Note $\gamma \overset{U}{\sim} 0 \Rightarrow \gamma \overset{U}{\approx} 0 \Rightarrow n(\gamma, a) = 0$.

Also $a \notin \Delta_i \Rightarrow n(c_i, a) = 0$ Then

$$n(\gamma^*, a) = \underbrace{n(\gamma, a)}_{0} + \sum (-m_i) \underbrace{n(c_i, a)}_{0} = 0.$$

[ii] if $a \in S$. Note that $n(c_i, a) = \begin{cases} 0 & \text{if } a \neq a_i \\ 1 & \text{if } a = a_i \end{cases}$

If $a = a_i \Rightarrow n(\gamma^*, a) = \underbrace{n(\gamma, a)}_{m_i} + (-m_i) \underbrace{n(c_i, a)}_{1} = m_i + (-m_i) = 0.$

If $a \neq a_i \ \forall i \Rightarrow n(\gamma, a) = 0$ by definition of the a_i's

$$\Rightarrow n(\gamma^*, a) = n(\gamma, a) + \sum (-m_i) \underbrace{n(c_i, a)}_{0} = 0.$$

Remarks

$\boxed{\text{i}}$ Proof of residue thm only requires $\gamma \overset{u}{\approx} 0$ not $\gamma \overset{u}{\sim} 0$. $\rightsquigarrow$ improvement of hypothesis.

$\boxed{\text{ii}}$ Residue Theorem $\Rightarrow$ Homology CIF for derivatives.

Let $\gamma \overset{u}{\approx} 0$. Apply the residue theorem: $S = \{a\}$.

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{k+1}}\, dz = n(\gamma, a) \operatorname*{Res}_{z=a} \frac{f(z)}{(z-a)^{k+1}}$$

$$= n(\gamma, a) \cdot \frac{f^{(k)}(a)}{k!}$$

(using Method 2 from last time)

Proof of Homology Cauchy's Theorem

- change notation $U \longleftrightarrow U^*$, $\gamma \longleftrightarrow \gamma^*$

- modify statement slightly

Theorem (Homology CIF)

$\gamma \overset{U}{\approx} 0$, $f : U \longrightarrow \mathbb{C}$ holomorphic, $a \in U \setminus \{\gamma\}$

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a}\, dz = n(\gamma, a)\, f(a).$$

Remark Using the above for $f^{new}(z) = f(z) \cdot (z-a)$, $f^{new}(a) = 0$

we obtain $\gamma \overset{U}{\approx} 0 \Rightarrow \int_{\gamma} f\, dz = 0$. This is Homology Cauchy.

Remark TFAE:

Homology CIF $\underset{\text{above}}{\Rightarrow}$ Homology Cauchy's theorem

$\underset{\text{page 4-5}}{\Rightarrow}$ Residue Theorem

$\underset{\text{page 6}}{\Rightarrow}$ Homology CIF for derivatives

$k = 0$

Theorem (Homology CIF / Conway IV. 5)

$\gamma \overset{U}{\approx} 0$, $f : U \longrightarrow \mathbb{C}$ holomorphic, $a \in U \setminus \{\gamma\}$

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - a} \, dw = n(\gamma, a) \, f(a) .$$

Rewriting.

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-a} \, dw = \frac{1}{2\pi i} \int_{\gamma} \frac{f(a)}{w-a} \, dw$$

$$\Longleftrightarrow \quad \int_{\gamma} \underbrace{\frac{f(w) - f(a)}{w - a}}_{\varphi(a, w)} \, dw = 0$$

Proof of Homology CIF

Auxiliary function $\varphi : U \times U \longrightarrow \mathbb{C}$

$$\varphi(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w}, & z \neq w. \\ f'(z), & z = w. \end{cases}$$

Want: $\displaystyle\int_\gamma \varphi(z,w)\,dw = 0 \quad \forall z \in U \quad (*)$

Apply $(*)$ to $z = a \in U \setminus \{\gamma\}$ to conclude Homology CIF.

Claims $\boxed{i}$ φ continuous in $U \times U$

$\boxed{ii}$ $z \longrightarrow \varphi(z,w)$ holomorphic $\forall w \in U$. fixed.

Proof of $\boxed{ii}$ This was explained in Lecture 10 as an application of Removable Singularity Theorem.

Proof of $\boxed{\text{L}}$ φ continuous in $\mathcal{U} \times \mathcal{U}$. Recall

$$\varphi(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w}, & z \neq w. \\[2ex] f'(z), & z = w. \end{cases}$$

- Continuity is clear at points where $z \neq w$.
- We show continuity at (a,a). We have

$$|\varphi(z,w) - \varphi(a,a)| = \left| \frac{1}{w-z} \int_z^w f'(t)\,dt - f'(a) \right|$$

$$= \frac{1}{|w-z|} \left| \int_z^w \left(f'(t) - f'(a)\right) dt \right|$$

$$\leq \sup_{t \in [z,w]} |f'(t) - f'(a)| < \varepsilon$$

if $z, w \in \Delta(a,\delta)$.

This holds in $\Delta(a,\delta)$ for some δ, because f' is continuous (in fact holomorphic).

Proof of (*) Want $\int_\gamma \varphi(z,w)\, dw = 0$ if $\gamma \overset{U}{\approx} 0$.

Question : How do we make use of $\gamma \overset{U}{\approx} 0$?

Answer. Define

$$V = \{ z \in \mathbb{C} \setminus \gamma,\ n(\gamma, z) = 0 \}.$$

- $U \cup V = \mathbb{C}$. (this is the only place where $\gamma \overset{U}{\approx} 0$ is used).

Indeed if $z \notin U \Rightarrow n(\gamma, z) = 0$ since $\gamma \overset{U}{\approx} 0$. Also $z \in \mathbb{C} \setminus \{\gamma\}$.

- V open. Indeed, by Lecture 6, V is union of components of $\mathbb{C} \setminus \{\gamma\} =$ open $\Rightarrow V$ open.

- V unbounded. In fact, by Lecture 6, $\exists\, R \gg 0$ with $\{ |z| > R \} \subseteq V$.

Define $h : \mathbb{C} \longrightarrow \mathbb{C}$

$$h(z) = \begin{cases} \int_\gamma \varphi(z,w)\, dw, & z \in U. \\ \int_\gamma \frac{f(w)}{w-z}\, dw, & z \in V \end{cases}$$

Claims $\boxed{a}$ h well - defined

$\boxed{b}$ h bounded, $\lim_{z \to \infty} h(z) = 0$

$\boxed{c}$ h entire

Conclusion By Liouville $\Rightarrow$ h constant $\overset{\boxed{b}}{\Rightarrow}$ $h \equiv 0$.

Thus if $z \in U \Rightarrow h(z) = \int_\gamma \varphi(z,w)\, dw = 0 \Rightarrow (*)$.

QED.

Proof of $\boxed{a}$ h well - defined. Take $z \in U \cap V$. We show

$$\int_\gamma \varphi(z,w)\, dw = \int_\gamma \frac{f(w)}{w-z}\, dw.$$

$$\Longleftrightarrow \int_\gamma \frac{f(z)}{w-z}\, dw = 0 \Longleftrightarrow f(z)\, n(\gamma, z) = 0$$ which is

true since $n(\gamma, z) = 0$ for $z \in V$.

Proof of $\boxed{b}$

Let $K > 0$ such that $\{\gamma\} \subseteq \Delta(0, K)$ by compactness.

We have $|w - z| \geq |z| - |w| \geq |z| - K$ if $w \in \{\gamma\}$.

If $R >> 0$, $|z| \geq R \Rightarrow z \in V$. then

$$|h(z)| = \left| \int_\gamma \frac{f(w)}{w - z} \, dw \right| \leq \underbrace{\text{length}(\gamma) \cdot \sup_{\{\gamma\}} |f| \cdot \frac{1}{|z| - K}}_{\downarrow \atop 0 \text{ as } z \to \infty}.$$

Since h is continuous by $\boxed{c}$ $\Rightarrow$ h bounded.

why?

- $\lim_{z \to \infty} h(z) = 0 \Rightarrow \exists \alpha$, $|h(z)| \leq 1$ if $|z| \geq \alpha$

  h continuous $\Rightarrow \exists M$, $|h(z)| \leq M$ if $|z| \leq \alpha$ $\Rightarrow$

$\Rightarrow |h| \leq \max(1, M)$.

Proof of $\boxed{c}$ h entire

Recall Conway Exercise IV. 2. 3. / HWK 3

Key statement $\varphi : \mathcal{U} \times \{\gamma\} \longrightarrow \mathbb{C}$

- φ continuous

- $z \longrightarrow \varphi(z, w)$ holomorphic $\forall\, w \in \{\gamma\}$.

Then $g(z) = \int_\gamma \varphi(z, w)\, dw$ holomorphic.

Proof See Solution Set 3.

Alternatively, let $\bar{R} \subseteq \mathcal{U}$. Then

$$\int_{\partial R} g\, dz = \int_{\partial R} \int_\gamma \varphi(z, w)\, dw\, dz$$

Fubini's theorem

φ continuous

$$= \int_\gamma \underbrace{\int_{\partial R} \varphi(z, w)\, dz}_{0}\, dw$$

Goursat's lemma or Cauchy

$$= \int_\gamma 0\, dw = 0$$

$\Rightarrow$ g admits a primitive in any disc $\Delta \subseteq \mathcal{U}$, $g = G'$

$\Rightarrow$ g holomorphic (G = holomorphic = ∞ - many times differentiable)

Back to $\boxed{C}$. Apply Key Statement to

- the set $\mathcal{U}$, for $\varphi = \phi \Rightarrow h$ holomorphic in $\mathcal{U}$
- the set V, for $\varphi(z,w) = \dfrac{f(w)}{w-z} : V \times \{\gamma\} \to \mathbb{C}$

$\Rightarrow h$ holomorphic in V.

Thus h is entire. QED.

2. Applications of the Residue theorem to real analysis

$$\frac{1}{2\pi i}\int_\gamma f\,dz = \sum_{s\in S} \operatorname{Res}(f,s)\cdot n(\gamma,s)\,, \quad \gamma \overset{u.}{\approx} 0$$

Applications

- [a] trigonometric functions
- [b] rational functions
- [c] Fourier integrals
- [d] logarithmic integrals
- [e] Mellin transforms

Poisson : "Je n'ai remarqué aucune intégrale qui ne fût pas déjà connue"