Math 220 A - Lecture 12

November 13 , 2023

Enhanced Cauchy's Theorem (Homology Cauchy)

We seek to prove a "homology" version of Cauchy:

Theorem $f: \mathcal{U}^* \longrightarrow \mathbb{C}$ holomorphic, $\mathcal{J}^* \approx 0$. Then $\int_{\mathcal{J}^*} f \, d_2 = 0$.

2> Conway IV. 5. 7.

 $\frac{Recall}{\chi} \approx 0 \quad means \quad n(\chi, a) = 0 \quad \forall a \notin U.$

Remark We will see next that

Homology Cauchy => Residue Thm.

Residue Theorem USE open connected, 5 discrete $\begin{array}{c} u \\ \gamma \sim 0, \quad \{\gamma\} \subseteq u \setminus s. \end{array}$ · f holomorphic in us, singularities at S. Then $\frac{1}{2\pi i} \int f d_2 = \sum_{z \in S} \operatorname{Res} (f, s) \cdot n (g, s).$

4, Conway V. 2.2

Proof of residue thrown We let f holomorphic in ULS, yro. We want $\frac{1}{2\pi i} \int f d_{z} = \sum_{s \in S} \operatorname{Res}(f, s) \cdot n(\gamma, s).$ We saw RHS is finite since $\begin{cases} j \in S: n(\gamma, s) \neq o \end{cases}$ is finite. Enumerale this set to be $\{a_1, \dots, a_k\}, m; = n(\gamma, a_i) \neq 0$. $Z_{ef} \Delta; be small disjoint discs mear a; \Delta; \leq 2L; \Delta; nS = fa; f.$ Define · u* = u 15 $\gamma^* = \gamma + \sum_{i=1}^{k} (-m_i) C_i \quad \text{where} \quad C_i = \partial \Delta_i$ (poortre orientation) $C_{laim} \gamma^* \approx 0$ Homology Cauchy for (u* x*) => \$ f d 2 = 0 $= \frac{1}{2\pi i} \int f d2 = \sum_{j=1}^{k} m_j \cdot \frac{1}{2\pi i} \int f d2$ $= \sum_{i=1}^{k} m_i Res (f, a_i) by toy example$ is last home. QED.

Proof of the claim Want n(y*a) = 0 if a & u*

 $\frac{1}{12} \quad if \quad a \neq u. \quad Nok \quad \gamma \sim o \quad = \ \gamma \approx o \quad = \ \circ (\gamma, a) = o.$ Also a & s; => n (c; a) = o Then $n(\gamma^*,a) = n(\gamma,a) + \sum (-m_i) n(c_i,a) = 0$ 0 O $[\alpha] if a \in S. Note that n(c, a) = \begin{cases} o & if a \neq a, \\ 1 & if a = a, \end{cases}$ $lf \ a = a; \implies n(\gamma, a) = n(\gamma, a) + (-m;) n(c;, a) = m; + (-m;) = 0.$ m : 1 If a = a; + i => n(r,a) = o by definition of the a;'s $\implies n(\gamma^{*}, a) = n(\gamma, a) + \sum (-m;) n(c; a) = 0.$

Remarks

101 Proof of residue the only requires 820 not

y no. ~ improvement. of hypothesis.

Residue Theorem => Homology CIF for derivatives.

Let & 20. Apply the residue theorem: S= fa }.

 $\frac{1}{2\pi i} \int \frac{f(2)}{(2-a)^{k+1}} dz = n(\gamma, a) Res \frac{f(2)}{2\pi i} = \frac{f(2)}{(2-a)^{k+1}}$

 $= n(\gamma, \alpha) \cdot \frac{f^{(k)}(\alpha)}{k!}$

(using Method 2 from last home)

Proof of Homology Caushy's Theorem

· change motation U and, 7 any*

· modify statement slightly

Theorem (Homology CIF)

γ≈0, f: u → a holomorphic, a ∈ U. (fr]

 $\frac{1}{2\pi i}\int \frac{f(z)}{z^2-a} dz = n(y,z) f(a).$

Remark Using the above for f (2) = f(2). (2-a) f (a)=0

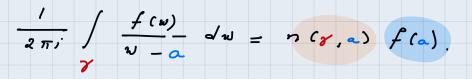
we obtain y = 0 =>) f d = 0. This is Homology Cauchy.

Remark TFAE:

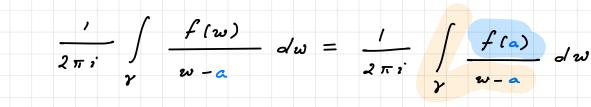
Homology CIF => Homology Cauchy 's theorem abor => Residue Theorem Page 4-5 => Homology CIF for derivatives page 6 **≭** =.0

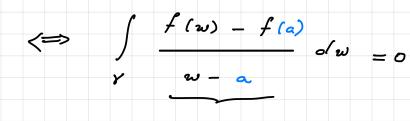
Theorem (Homology CIF / Conway IV. 5)

 $\gamma \approx 0$, $f: u \longrightarrow a$ holomorphic, $a \in U \setminus \{r\}$









y (a, w)

Proof of Homology CIF

Auxiliary function 9: Ux u - C

 $\varphi(z,w) = \begin{cases} \frac{f(z) - f(w)}{z - w}, & z \neq w. \\ f'(z), & z \neq w. \end{cases}$

Want: $\int \varphi(z, w) dw = 0 \qquad \forall z \in u \quad (*)$

Apply (*) to g = a GUN } to conclude Homology CIF

Claims II y continuous in uxu

Proof of 1001 This was explained in Lecture 10 as

an application of Removable Singularity Theorem.

y continuous in UXU. Recall Proof of II

 $\varphi(2, w) = \begin{cases} \frac{f(2) - f(w)}{2 - w}, & 2 \neq w \\ f'(2) & , 2 \neq w \end{cases}$

· Continuity is clear at points where 2 = w.

· We show continuity at (a,a). We have

 $|\varphi(2,w) - \varphi(a,a)| = \left| \frac{1}{w-2} \int_{2}^{w} f'(t) dt - f'(a) \right|$

 $=\frac{1}{|w-z|} \int_{2}^{w} (f'(t) - f'(a)) dt \Big|$

 $\leq \sup |f'(t) - f'(a)| < \varepsilon$ $t \in [2, w] \qquad if 2, w \in \Delta(a, S).$

This holds in \$ (9,8) for some 8, because f'is

continuous (in fact holomorphic).

 $\frac{P_{roof} \circ f(x)}{\gamma} \quad \text{Want} \quad \int \varphi(z,w) \, dw = 0 \quad \text{if } \gamma \approx 0.$ Question: How do we make use of y 20? Answer Define $V = \begin{cases} 2 \in \mathcal{C} \setminus \gamma, \quad n(\gamma, 2) = 0 \end{cases}$ • $U \cup V = T$. C this is the only place where $y \approx 0$ is used)_ Indeed if $2 \notin (u =) n(y, 2) = 0$ since $\gamma \approx 0$. Also $2 \in \mathcal{I} \setminus \{r\}$. · V open . Indeed, by Leotur 6, V is uncon of components of Eigr} = open => V open. · V un bounded. In fact, by Zecture 6, JR>>0 with $\{1_2 | > R \} \subseteq V$.

Define h: a - a $-h(g) = \begin{cases} \int \varphi(g,w) dw , g \in \mathcal{U}. \\ r & \\ \int \frac{f(w)}{w-g} dw , g \in \mathcal{V} \\ r & \\ \hline w - g & \\ \end{array}$ 1 h well - defined Claim 3 $\begin{array}{c} \hline b \end{array} \\ \hline b \bigg \\ b \bigg \\ \hline b \bigg \\ b \bigg \\ b \bigg \\ \hline b \bigg \\ b \bigg \\ \hline b \bigg \\ b \bigg$ ICT h entire Conclusion By Jiouville => h constant => h =0. Proof of A will - defined. Take ZE UnV. We show $\int \varphi(x,w) dw = \int \frac{\varphi(w)}{w-x} dw.$ $\begin{array}{c} \langle = \rangle \int \frac{f(z)}{w-z} dw = 0 \quad \langle = \rangle \quad f(z) \quad n(z,z) = 0 \quad which is \\ \end{array}$ the since n(x,z) =0 for z e V.

Proof of 151

Let K70 such that {7} = D(0,K) by compactness

We have 12-212121-12121-K if we 383.

17 R>>0, 1212R => 2 EV. Then

 $|h(z)| = \left| \int \frac{f(w)}{w-z} dw \right| \leq length(y) \cdot sup(f) \cdot \frac{1}{121-k}$

Since h is continuous by [] => h bounded.

why?

=> 1h1 < max (1, M).

6 0 as 2 →∞.

Proof of [] hentre Recall Conwoy Exercise IV. 2. 3. / HWK3

Key statement up: ux fr} -> c

· y continuous

· 2 - y(2,2) tolomophic + we for J.

Then $g(z) = \int \psi(z,w) dw$ holomorphic.

Proof See Solution Set 3.

Alternatively, let R = U. Then

=> g admits a primitive in any disc $\Delta \subseteq \mathcal{U}$, g = c'

=> g holomorphic (= holomorphic = 00 - many times differentiable)

Back to C. Apply Key Statement to

· the set u, for y = of => h holomorphic in U

• the set V, for $\psi(z, w) = \frac{f(w)}{w-z} : V \times \{r\} \to C$

=> h holomorphic in V.



2. Applications of the Residue Theorem to real analysis

 $\frac{1}{2\pi}, \int f d_{z} = \sum_{s \in S} \mathcal{R}_{es}(f, s), n(\gamma, s), \gamma \approx 0$ $j \in S$

Applications [] trigonome tric functions 5 rational functions 5 Jounier integrals

1 logan thmic integrals

Mellin transforms

Poisson: "Je n'ai remargue aucune intégrale qui

me fuit pas déjà connue