Math 220 A – Lecture 13

November 15, 2023

Applications of the Residue theorem to real analysis

$$\frac{1}{2\pi i}\int_\gamma f\,dz = \sum_{s\in S} \operatorname{Res}(f,s)\cdot n(\gamma,s)\,, \quad \gamma \overset{u.}{\approx} 0$$

Applications

[a] trigonometric functions

[b] rational functions

[c] Fourier integrals

[d] logarithmic integrals

[e] Mellin transforms

[a] Trigonometric integrals

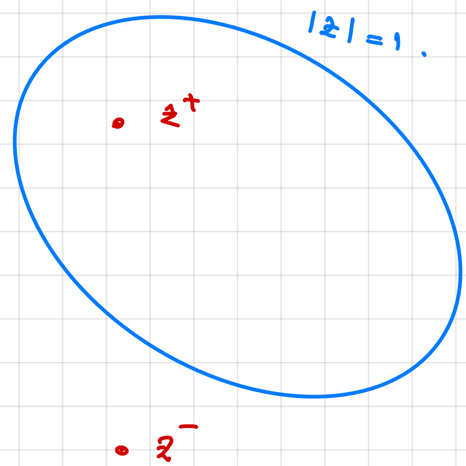
Example $a > 1$, $a \in \mathbb{R}$. $\quad I = \displaystyle\int_0^{2\pi} \frac{dt}{a + \sin t}$.

$$z = e^{it} \implies \frac{dz}{iz} = dt$$

$$\sin t = \frac{z - z^{-1}}{2i}.$$

By substitution, we find

$$I = \int_{|z|=1} \frac{2\, dz}{z^2 - 1 + 2aiz}.$$



poles $z^2 - 1 + 2aiz = 0$

$$\implies z^{\pm} = -ai \pm i\sqrt{a^2 - 1}$$

Note $|z^+| < 1$, $|z^-| > 1$. Thus

Method 1

$$I = 2\pi i \operatorname{Res}(f, z^+) = 2\pi i \cdot \frac{2}{(z^2 - 1 + 2aiz)'}\Bigg|_{z = z^+}$$

$$= 2\pi i \cdot \frac{2}{2z + 2ai}\Bigg|_{z = z^+} = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

$\boxed{b}$ Rational functions

$$I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx.,$$

Require : (1) Q has no zeros on the real axis

(2) $\deg P - \deg Q \leq -2.$

Claim : The integral converges absolutely

Write $f(x) = \dfrac{P(x)}{Q(x)}$.

By (2) $\Rightarrow \lim\limits_{|x| \to \infty} x^2 f(x) = \alpha < \infty \Rightarrow \exists\, R > 0$ with

$$|f(x)| < \frac{\alpha+1}{x^2} \quad \text{for } |x| > R. \qquad (*)$$

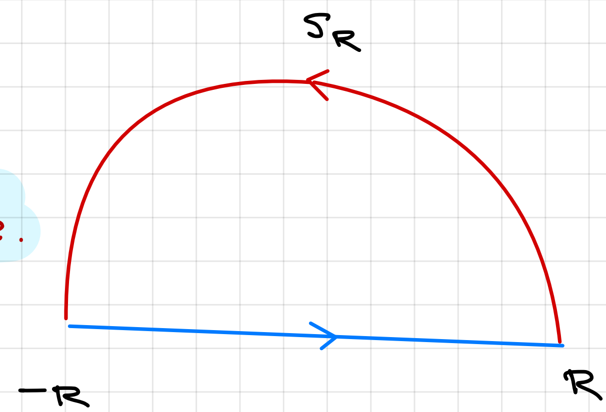
By the comparison test $\Rightarrow \displaystyle\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$ Q.E.D.

Conclusion $\displaystyle I = \lim_{R \to \infty} \int_{-R}^{R} f(x)\, dx.$

Strategy

[I] $f(z) = \dfrac{P(z)}{Q(z)}$

$$\gamma_R = [-R, R] \cup S_R .$$



[II] Residue theorem

$$\int_{-R}^{R} f(x)\,dx + \int_{S_R} f\,dz = \int_{\gamma_R} f\,dz = 2\pi i \sum_{\substack{a_j \in \mathbb{H}^+ \\ |a_j| < R.}} \operatorname{Res}(f, a_j).$$

[III] Make $R \to \infty$. Show $\displaystyle \lim_{R\to\infty} \int_{S_R} f\,dz = 0$

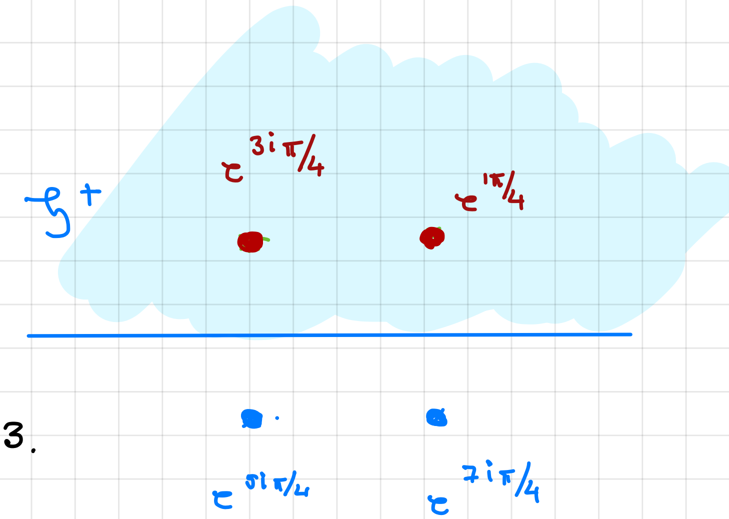
$$\left| \int_{S_R} f\,dz \right| \leq \pi R \cdot \frac{\alpha + 1}{R^2} \longrightarrow 0 \text{ as } R \longrightarrow \infty. \text{ using } (*).$$

From [II], we obtain

Conclusion

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{a_j \in \mathbb{H}^+} \operatorname{Res}\left(\frac{P}{Q}, a_j\right).$$

Example $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$



Poles at $z^4+1=0$

$$\Rightarrow \quad z_k = e^{\frac{\pi i}{4}(2k+1)}, \quad k=0,1,2,3.$$

Only $e^{\pi i/4}, \ e^{3\pi i/4} \in \mathfrak{H}^+$.

By Method 1,

$$\operatorname*{Res}_{z=z_k} \frac{1}{z^4+1} = \frac{1}{4z^3}\Big/_{z=z_k} = \frac{1}{4z_k^3} = -\frac{z_k}{4}.$$

Thus

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\pi i \left(\operatorname*{Res}_{z=e^{\pi i/4}} \frac{1}{z^4+1} + \operatorname*{Res}_{z=e^{3\pi i/4}} \frac{1}{z^4+1} \right)$$

$$= 2\pi i \left(-\frac{1}{4} e^{\pi i/4} - \frac{1}{4} e^{3\pi i/4} \right)$$

$$= \frac{\pi}{\sqrt{2}}.$$

[c] Fourier Integrals I

$$I = \int_{-\infty}^{\infty} f(x)\, e^{ix}\, dx \quad \text{(use upper half plane)}$$

$$I = \int_{-\infty}^{\infty} f(x)\, e^{-ix}\, dx \quad \text{(use lower half plane)}$$

Require

(1) f extends meromorphically to $\mathfrak{H}^+$

(2) no poles on the real axis.

(3) $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$

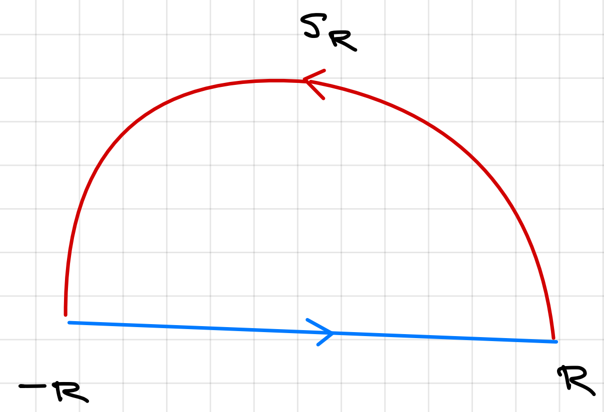
Convergence: By (3), $\int_{-\infty}^{\infty} f(x)\, e^{ix}\, dx$ converges absolutely

Thus

$$I = \lim_{R \to \infty} \int_{-R}^{R} f(x)\, e^{ix}\, dx.$$

Strategy Use the same contour

$$\gamma_R = [-R, R] \cup S_R.$$



By the residue theorem

$$\int_{-R}^{R} f(x) e^{ix}\, dx + \int_{S_R} f(z) e^{iz}\, dz = 2\pi i \sum_{\substack{z = a_j \\ |a_j| < R \\ a_j \in \mathcal{H}^+}} \operatorname{Res}\left(f(z) e^{iz}\right)$$

Make $R \to \infty$. Assume moreover

$$(4) \quad \lim_{\substack{z \to \infty \\ z \in \overline{\mathcal{H}}^+}} f(z) = 0.$$

The next lemma shows $\displaystyle\lim_{R \to \infty} \int_{S_R} f(z) e^{iz}\, dz = 0$

Conclusion

$$\int_{-\infty}^{\infty} f(x) e^{ix}\, dx = 2\pi i \sum_{a_j \in \mathcal{H}^+} \operatorname{Res}\left(f(z) e^{iz}, a_j\right).$$

Lemma If $\lim\limits_{\substack{z\to\infty \\ z\in\overline{\mathbb{H}}^{+}}} |f(z)| = 0$ then

$$\lim_{R\to\infty} \int_{S_R} f(z)\, e^{iz}\, dz = 0$$

Proof Write $z = Re^{it}$, $0 \le t \le \pi$.

$$M_R = \sup_{z\in S_R} |f(z)| \ , \quad M_R \longrightarrow 0 \quad \text{as } R \longrightarrow \infty$$

$$\left| \int_{S_R} f(z)\, e^{iz}\, dz \right| = \left| \int_0^{\pi} f(Re^{it})\, e^{iRe^{it}} \cdot iRe^{it}\, dt \right|$$

$$\le \int_0^{\pi} M_R \cdot \left| e^{iRe^{it}} \right| \cdot R\, dt$$

$$= \int_0^{\pi} M_R \left| e^{iR(\cos t + i\sin t)} \right| R\, dt$$

$$= \int_0^{\pi} R M_R \left| e^{iR\cos t}\, e^{-R\sin t} \right| dt$$

$$= \int_0^{\pi} R M_R\, e^{-R\sin t}\, dt$$

$$= 2\int_0^{\pi/2} R M_R\, e^{-R\sin t}\, dt$$

Claim

$$\le 2\int_0^{\pi/2} R M_R\, e^{-R\cdot\frac{2}{\pi}t} = \pi M_R \left(1 - e^{-R}\right) \longrightarrow 0$$

as $R \longrightarrow \infty$.

Claim $\frac{2}{\pi} \le \frac{\sin t}{t} \quad \forall\ t \in \left(0, \frac{\pi}{2}\right]$

Proof $f(t) = \frac{\sin t}{t}$.

$$f(\pi/2) = \frac{2}{\pi}$$

We show f is decreasing. Then $f(t) \ge f\left(\frac{\pi}{2}\right) = \frac{2}{\pi} \Rightarrow$

$\Rightarrow \frac{\sin t}{t} \ge \frac{2}{\pi}$

To this end, compute $f'(t) = \dfrac{t\cos t - \sin t}{t^2} \le 0$

$$\Longleftrightarrow \quad t \cos t \le \sin t$$

$$\Longleftarrow \quad \tan t - t \ge 0.$$

Let

$$g(t) = \tan t - t, \quad g(0) = 0$$

We compute $g'(t) = \dfrac{1}{\cos^2 t} - 1 \ge 0 \Rightarrow g \nearrow \Rightarrow$

$\Rightarrow g(t) \ge g(0) = 0$ as needed. QED

Example $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = \operatorname{Re} I. = \frac{\pi}{e}.$

Let $f(z) = \dfrac{1}{1+z^2}$, $z = i$ is the only pole in $\mathcal{H}^+$.

Method 1

$$I = \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{1+z^2}\right)\Big/_{z=i} =$$

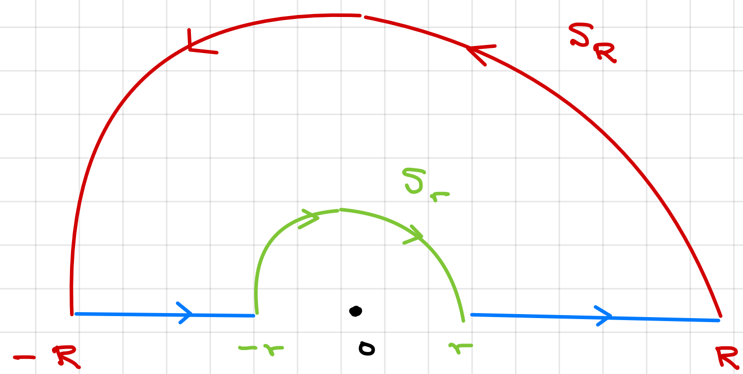
$$= 2\pi i \cdot \frac{e^{iz}}{2z}\Big/_{z=i}$$

$$= \cancel{2}\pi \cancel{i} \cdot \frac{e^{-1}}{\cancel{2}\cancel{i}} = \frac{\pi}{e}.$$

Fourier Integrals – Part II – Poles on the real axis

Example $$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

- issues at 0 & ∞.
- $$I = \lim_{\substack{r\to 0\\ R\to\infty}} \int_r^R \frac{\sin x}{x}\,dx$$



Strategy
- $$f(z) = \frac{e^{iz}}{z}$$
- $$\gamma = S_R + [-R,-r] + (-S_r) + [r,R]$$

$$0 = \int_\gamma f\,dz = \int_{S_R} f\,dz - \int_{S_r} f\,dz + \int_r^R \frac{e^{iz}}{z}\,dz + \int_{-R}^{-r} \frac{e^{iz}}{z}\,dz$$

$$= \int_{S_R} f\,dz - \int_{S_r} f\,dz + \int_r^R \frac{e^{iz}}{z} - \frac{e^{-iz}}{z}\,dz$$

$$= \int_{S_R} f\,dz - \int_{S_r} f\,dz + \int_r^R 2i\,\frac{\sin z}{z}\,dz$$

Make $r \to 0$, $R \to \infty$. By the claim:

$$0 = 0 - i\pi + 2i\int_0^\infty \frac{\sin x}{x}\,dx \implies \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Claims $\boxed{a}$ $\displaystyle\lim_{R\to\infty} \int_{S_R} \frac{e^{iz}}{z}\, dz = 0$

$\boxed{b}$ $\displaystyle\lim_{r\to 0} \int_{S_r} \frac{e^{iz}}{z}\, dz = i\pi$

Claim $\boxed{a}$ follows from the previous Lemma applied to

$f(z) = \frac{1}{z}$.

Part $\boxed{b}$ uses the next lemma for $g(z) = \frac{1}{z}$

Lemma Let g have simple pole at 0. Then

$$\lim_{r\to 0} \int_{S_r} g(z)\, e^{iz}\, dz = \pi i \operatorname{Res}(g, 0).$$

Proof Since g has a simple pole at 0, write

$$g(z) = \frac{\alpha}{z} + \underbrace{G(z)}_{\text{Taylor series}}$$

$\alpha = \operatorname{Res}(g, 0)$, G holomorphic near 0

$$e^{iz} = 1 + \underbrace{z F(z)}_{\text{Taylor}}, \quad F \text{ holomorphic near } 0$$

$$e^{iz} g(z) = \left(\frac{\alpha}{z} + G\right)\left(1 + zF\right) = \frac{\alpha}{z} + H,$$

$H = G + zFG + \alpha F$ holomorphic near 0

$\Longrightarrow$ H bounded near 0. $\Longrightarrow$ $\exists M, \delta$: $|H(z)| \le M$ if $|z| \le \delta$.

Compute $$\int_{S_r} e^{iz} g(z)\, dz = \int_{S_r} \frac{\alpha}{z} + H \, dz.$$

Note $$\alpha \int_{S_r} \frac{dz}{z} = \alpha \int_0^{\pi} \frac{d(re^{it})}{re^{it}} = \alpha \int_0^{\pi} i\, dt = \pi i \alpha$$

$$\left| \int_{S_r} H\, dz \right| \le M \cdot \pi r \longrightarrow 0 \quad \text{as } r \longrightarrow 0.$$

Thus $$\int_{S_r} e^{iz} g(z)\, dz \longrightarrow \pi i \alpha \quad \text{as } r \longrightarrow 0, \text{ as claimed.}$$