# Math 220 A — Lecture 14

November 20, 2023

**Lemma** Let $g$ have simple pole at $0$. Then

needed last time

$$\lim_{r \to 0} \int_{S_r} g(z)\, e^{iz}\, dz = \pi i \operatorname{Res}(g, 0).$$

**Proof** Since $g$ has a simple pole at $0$, write near $0$:

Taylor series

$$g(z) = \frac{\alpha}{z} + G(z) \qquad \alpha = \operatorname{Res}(g, 0), \; G \text{ holomorphic}$$

Taylor

$$e^{iz} = 1 + z F(z), \qquad F \text{ holomorphic near } 0$$

$$e^{iz} g(z) = \left(\frac{\alpha}{z} + G\right)\left(1 + zF\right) = \frac{\alpha}{z} + H,$$

$H = G + zFG + \alpha F$ holomorphic near $0$

$\Rightarrow H$ bounded near $0$. $\Rightarrow \exists M, \delta :$ $|H(z)| \le M$ if $|z| \le \delta$.

Compute $$\int_{S_r} e^{iz} g(z)\, dz = \int_{S_r} \frac{\alpha}{z} + H \, dz.$$

Note $$\alpha \int_{S_r} \frac{dz}{z} = \alpha \int_0^{\pi} \frac{d(re^{it})}{re^{it}} = \alpha \int_0^{\pi} i\, dt = \pi i \alpha$$

$$\left| \int_{S_r} H\, dz \right| \le M \cdot \pi r \longrightarrow 0 \text{ as } r \longrightarrow 0.$$

Thus $\displaystyle\int_{S_r} e^{iz} g(z)\, dz \longrightarrow \pi i \alpha$ as $r \longrightarrow 0$, as claimed.

# Applications of the Residue theorem to real analysis

- a) trigonometric functions
- b) rational functions
- c) Fourier integrals
- d) logarithmic integrals
- e) Mellin transforms

## d) Logarithmic integrals

Conway V. 2. 10.

$$\int_0^\infty R(x) \log x \, dx$$

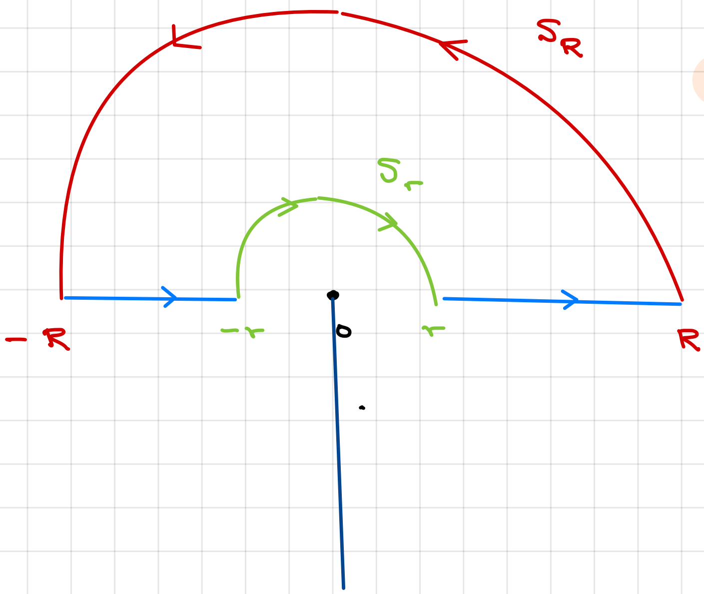
$R$ = even rational function without real poles

**Example** $R(x) = \frac{1}{1+x^2} \implies \int_0^\infty \frac{\log x}{1+x^2}\, dx = 0$

**HWK** $R(x) = \frac{1}{(1+x^2)^2} \implies \int_0^\infty \frac{\log x}{(1+x^2)^2}\, dx.$

**Issues**: – logarithm undefined at 0 (use circle $S_r$)

– holomorphic extension for logarithm




Requires branch cut!

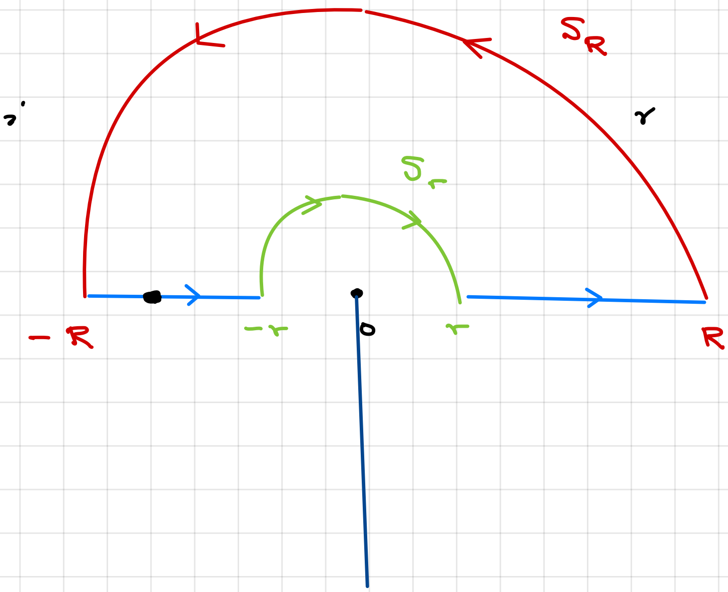
Define for $z = re^{it}$

$\ell(z) = \log r + it$

$-\frac{\pi}{2} < t < \frac{3\pi}{2}$

- $\gamma = S_R + [-R, -r] + (-S_r) + [r, R]$

- $f(z) = \dfrac{\ell(z)}{1+z^2}$ has pole at $i$




## Residue theorem

Method 1

$$\operatorname{Res}(f, i) = \operatorname*{Res}_{z=i} \frac{\ell(z)}{1+z^2} = \left.\frac{\ell(z)}{2z}\right|_{z=i} = \frac{i\pi/2}{2i} = \frac{\pi}{4}.$$

Residue thm:

$$\int_\gamma f\,dz = 2\pi i \operatorname{Res}(f, i) = i\,\frac{\pi^2}{2}. \qquad (*).$$

$$\int_{S_R} f\,dz - \int_{S_r} f\,dz + \int_r^R f(x)\,dx + \int_{-R}^{-r} f(x)\,dx.$$

We make $r \longrightarrow 0$, $R \longrightarrow \infty$.

## Segment integrals

see the definition of $\ell$

$$\int_r^R \frac{\ell(x)}{1+x^2}\,dx + \int_{-R}^{-r} \frac{\ell(x)}{1+x^2}\,dx = \int_r^R \frac{\log x}{1+x^2}\,dx + \int_{-R}^{-r} \frac{\log|x| + i\pi}{1+x^2}\,dx$$

$$= 2\int_r^R \frac{\log x}{1+x^2}\,dx + i\pi \int_{-R}^{-r} \frac{dx}{1+x^2}$$

$$\xrightarrow[R\to\infty]{r\to 0} 2\int_0^\infty \frac{\log x}{1+x^2}\,dx + i\pi \arctan x \Big|_{x=-\pi}^{x=0}$$

$$= 2\int_0^\infty \frac{\log x}{1+x^2}\,dx + i\pi \cdot \frac{\pi}{2}$$

**Claim**

$$\lim_{\substack{\rho\to 0\\ \rho\to\infty}} \int_{S_\rho} \frac{\ell(z)}{1+z^2}\,dz = 0.$$

**Conclusion** From $(*)$ we get as $r\to 0$, $R\to\infty$:

$$\frac{i\pi^2}{2} = 2\int_0^\infty \frac{\log x}{1+x^2}\,dx + i\pi \cdot \frac{\pi}{2}$$

$$\Longrightarrow \int_0^\infty \frac{\log x}{1+x^2}\,dx = 0$$

Proof of the claim $\quad z = \rho e^{it}, \quad 0 \le t \le \pi$

$$\left| \int_{S_\rho} \frac{\ell(z)}{1+z^2}\, dz \right| = \left| \int_0^\pi \frac{\log\rho + it}{1+\rho^2 e^{2it}} \cdot \rho e^{it}\, i\, dt \right|$$

$$\le \int_0^\pi \frac{|\log\rho| + \pi}{|1+\rho^2 e^{2it}|} \cdot \rho\, dt$$

$$\le \int_0^\pi \frac{|\log\rho| + \pi}{|\rho^2 - 1|} \cdot \rho\, dt$$

$$= \pi \cdot \frac{\rho\,|\log\rho|}{|\rho^2-1|} + \pi^2 \cdot \frac{\rho}{|\rho^2-1|} \longrightarrow 0.$$

As $\rho \longrightarrow \infty$, $\dfrac{\rho \log\rho}{\rho^2-1}$ and $\dfrac{\rho}{\rho^2-1} \longrightarrow 0$.

As $\rho \longrightarrow 0$, the same is true.

The only term that requires justification is

$\rho \log\rho = -\dfrac{w}{e^w} \longrightarrow 0$ as $w \to \infty$, where $\rho = e^{-w}$, $\rho \longrightarrow 0$.

# Applications of the Residue theorem to real analysis

- a) trigonometric functions
- b) rational functions
- c) Fourier integrals
- d) logarithmic integrals
- e) Mellin transforms

## Mellin transforms

Conway V. 2.12.

$$\int_0^\infty \frac{R(x)}{x^\alpha}\,dx \quad , \quad 0 < \alpha < 1$$

$R$ = rational function, no poles on positive real axis

Useful in prime counting & number theory

**Example** $\quad R(x) = \dfrac{1}{x+1} \implies \displaystyle\int_0^\infty \frac{dx}{x^\alpha (x+1)} = \frac{\pi}{\sin \pi\alpha}$

(next)

**Homework** $\quad R(x) = \dfrac{1}{x^n+1} \implies \displaystyle\int_0^\infty \frac{dx}{x^\alpha (x^n+1)}$

## Remark

[i] Fourier transform

$$f \rightsquigarrow \mathcal{F}f(s) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x s}\, dx$$

[ii] Laplace transform

$$f \rightsquigarrow \mathcal{L}f(s) = \int_{0}^{\infty} f(x)\, e^{-sx}\, dx$$

[iii] Mellin transform

$$f \rightsquigarrow \mathcal{M}f(s) = \int_{0}^{\infty} f(x)\, \underbrace{x^{s-1}}_{x^{-\alpha}}\, dx$$

$x^{-\alpha}$ on previous page

## Remark (will not use)

The Mellin transform of $f(x) = e^{-x}$ is the $\Gamma$-function

$$\Gamma(s) = \int_{0}^{\infty} e^{-x} x^{s-1}\, dx$$

Hjalmar Mellin (1854 – 1933)

Finnish mathematician

**Example** $R(x) = \dfrac{1}{x+1} \Rightarrow \boxed{\displaystyle\int_0^\infty \frac{dx}{x^\alpha (x+1)} = \frac{\pi}{\sin \pi\alpha}}$

for $0 < \alpha < 1$

**Convergence** uses $0 < \alpha < 1$.

- $0 < x < 1$ : $\displaystyle\int_0^1 \frac{dx}{x^\alpha (x+1)} < \int_0^1 \frac{dx}{x^\alpha} = \frac{x^{-\alpha+1}}{-\alpha+1}\Bigg/_{x=0}^{x=1} < \infty$

- $1 < x < \infty$ : $\displaystyle\int_1^\infty \frac{dx}{x^\alpha (x+1)} < \int_1^\infty \frac{dx}{x^{\alpha+1}} = \frac{x^{-\alpha}}{-\alpha}\Bigg/_{x=1}^{x=\infty} < \infty$

Therefore

$$I = \int_0^\infty \frac{dx}{x^\alpha (x+1)} = \lim_{\substack{r \to 0 \\ R \to \infty}} \int_r^R \frac{dx}{x^\alpha (x+1)} . < \infty$$

$$I = \int_0^\infty \frac{dx}{x^\alpha (x+1)}$$

Question : [a] What function?

[b] What contour?

Issues [i] extend $x^\alpha$ holomorphically

$$z^\alpha = \exp(\alpha\, \ell(z)) \rightsquigarrow \text{branch cut along } [0, \infty).$$

For $z = re^{it}$, $\ell(z) = \log r + it$, $0 < t < 2\pi$

[ii] pole at $0$ — use $C_r$ to isolate the pole

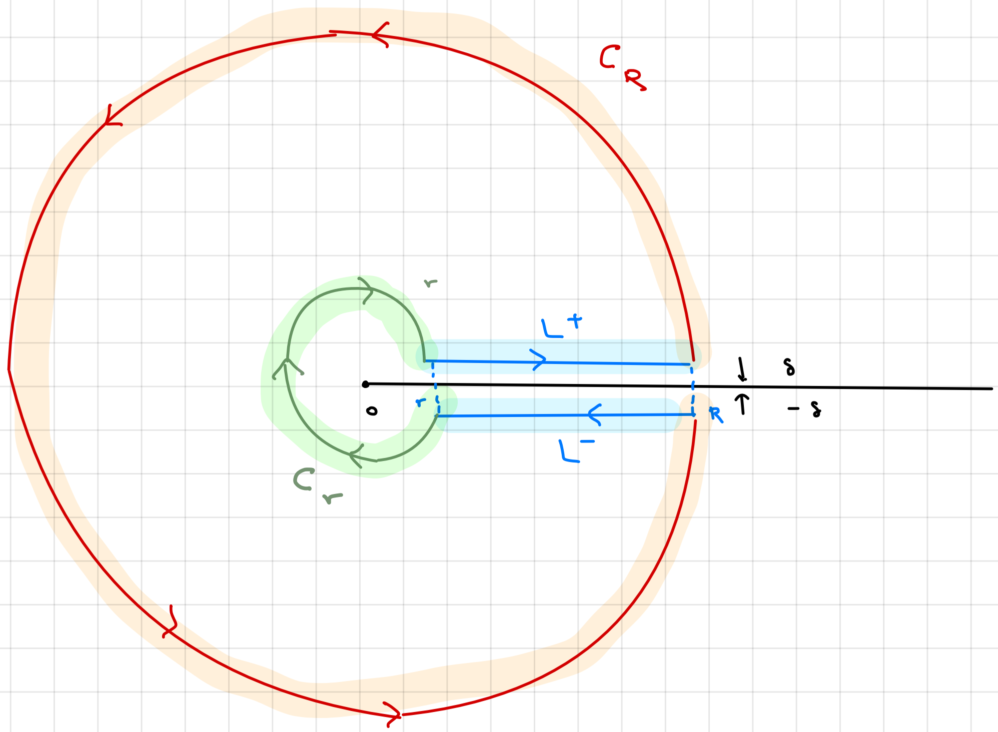
Remark It is precisely the fact that we cut along $[0, \infty)$.

($=$ domain of integration) that allows us to calculate $I$.

Before, we were cutting away from domain of integration

Solutions (a) $f(z) = \dfrac{1}{z^\alpha (z+1)}$

(b) $\gamma$ = key-hole contour




$$\gamma = C_R + (-L^-) + (-C_r) + L^+$$

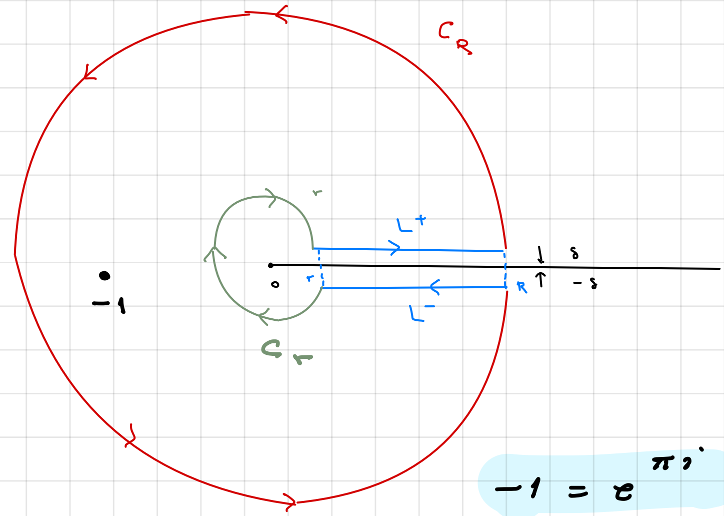
$C_R$ = circle from $R + i\delta$ to $R - i\delta$

$C_r$ = circle from $r + i\delta$ to $r - i\delta$

$L^\pm$ = segment from $r \pm i\delta$ to $R \pm i\delta$.

## Residue theorem

$$f(z) = \frac{1}{z^{\alpha}(z+1)} \quad . \quad \text{pole at } -1.$$




$-1 = e^{\pi i}$

Method 1

$$\operatorname{Res}(f,-1) = \operatorname*{Res}_{z=-1} \frac{\frac{1}{z^{\alpha}}}{z+1} = \frac{1}{(-1)^{\alpha}} = \frac{1}{e^{\pi i \alpha}} = e^{-\pi i \alpha}$$

---

$$\int_{\gamma} f\,dz = 2\pi i \operatorname{Res}(f,-1) = 2\pi i \exp(-\alpha\pi i).$$

$$\int_{C_R} f\,dz - \int_{C_r} f\,dz + \int_{L^+} f\,dz - \int_{L^-} f\,dz \qquad (R).$$

Make $r \to 0$, $R \to \infty$, $\delta \to 0$.

---

## Claims

$\boxed{a}$ $$\lim_{\substack{\delta \to 0 \\ \rho \to 0 \text{ or} \\ \rho \to \infty}} \int_{C_\rho} \frac{dz}{z^\alpha (z+1)} = 0$$

$\boxed{b}$ $$\lim_{\substack{\delta \to 0 \\ r \to 0 \\ R \to \infty}} \int_{L^+} \frac{dz}{z^\alpha (z+1)} = I$$

$\boxed{c}$ $$\lim_{\substack{\delta \to 0 \\ r \to 0 \\ R \to \infty}} \int_{L^-} \frac{dz}{z^\alpha (z+1)} = e^{-2\pi i\alpha} I$$

**Conclude** In $(R)$ make $\delta \to 0$, $r \to 0$, $R \to \infty$:

$$0 - 0 + I - e^{-2\pi i\alpha} I = e^{-\pi i\alpha} \cdot 2\pi i$$

$$I = \frac{2\pi i \, e^{-\pi i\alpha}}{1 - e^{-2\pi i\alpha}} = \frac{2\pi i}{e^{\pi i\alpha} - e^{-\pi i\alpha}} = \frac{\pi}{\sin \pi\alpha}.$$

## Proof of $\boxed{a}$

$$\left| \iint_{C_\rho} \frac{dz}{z^\alpha (z+1)} \right| \le 2\pi \rho^* \cdot \frac{1}{\rho^\alpha |\rho - 1|} \longrightarrow 0$$

as $\delta \to 0$ and $\rho \longrightarrow 0$ or $\rho \longrightarrow \infty$. because $0 < \alpha < 1$.

Here $\rho^* = \sqrt{\rho^2 + \delta^2}$ = radius of $C_\rho$

## Proof of $\boxed{b}$

$g(z) = \dfrac{1}{z^\alpha}$, $L^+ = \{ t + i\delta : r \le t \le R \}$.

$$\lim_{\delta \to 0} \int_{L^+} \frac{g(z)}{z+1} dz \overset{(+)}{=} \int_r^R \frac{t^{-\alpha}}{t+1} dt. \longrightarrow I.$$

as $r \longrightarrow 0$
$R \longrightarrow \infty$.

Why $(+)$?

$$\int_{L^+} \frac{g(z)}{z+1} dz = \int_r^R \frac{g(t+i\delta)}{1+t+i\delta} dt \xrightarrow{\delta \to 0} \int_r^R \frac{t^{-\alpha}}{1+t} dt.$$

Define

$$G(t,\delta) = \begin{cases} \dfrac{g(t+i\delta)}{1+t+i\delta} - \dfrac{t^{-\alpha}}{1+t}, & \delta \neq 0. \\ 0, & \delta = 0. \end{cases}$$

$r \le t \le R$, $0 \le \delta \le 1$.

$G$ continuous (uniformly). Given any $\varepsilon$, $\exists\ \tau > 0$ such that if

$$|\delta - 0| < \tau,\quad |t - t| < \tau \implies |G(t,\delta) - \underbrace{G(t,0)}_{0}| < \varepsilon.$$

$$\implies |G(t,\delta)| < \varepsilon. \implies \left| \int_r^R G(t,\delta)\, dt \right| \leq (R-r)\cdot\varepsilon \text{ as } |\delta| < \tau.$$

## Proof of [c]

**Difference** $\quad g(t - i\delta) \longrightarrow t^{-\alpha} \cdot e^{-2\pi i \alpha}$.

The rest of the proof is the same as [b].

Indeed $g(t - i\delta) = (t - i\delta)^{-\alpha} = \exp(-\alpha\, \ell(t - i\delta))$

$$\xrightarrow{\delta \to 0} \exp(-\alpha \log t - 2\pi i \alpha) = t^{-\alpha} \cdot e^{-2\pi i \alpha}.$$

This explains the extra factor $e^{-2\pi i \alpha}$ in the answer to [c].