Math 220 A — Lecture 15

November 22, 2023

Residues at ∞ & Shadows of Riemann Surfaces

[A]. Topology on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

recall from

Lecture 3

- basic neighborhood of ∞

$$\mathcal{U} = \{\infty\} \cup \{|z| > R\} \text{ for some } R.$$

- $\hat{\mathbb{C}}$ is a topological space

- $\hat{\mathbb{C}}$ compact, $\hat{\mathbb{C}} \cong S^2 \subseteq \mathbb{R}^3$

$\cong$: homeomorphism

Remark $f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$, $\quad z \longrightarrow \dfrac{1}{z}$, $\quad f(0) = \infty$

$f(\infty) = 0$

(punctured) neighborhoods of 0 $\qquad |z| < r$

$$z \longrightarrow \frac{1}{z}.$$

$$\left|\frac{1}{z}\right| > \frac{1}{r}$$

(punctured) neighborhoods of ∞

$\boxed{B.}$ Singularities & Residues at ∞

Recall Conway V.1.13 — Pset 5

If $f : \{|z| > R\} \longrightarrow \mathbb{C}$ holomorphic

$\Longrightarrow$ ∞ is isolated singularity

Types

$\boxed{i}$ removable

$\boxed{ii}$ pole $\Longrightarrow$ $g(z) = f\left(\frac{1}{z}\right)$.

$\boxed{iii}$ essential Inspect singularity at 0.

Example $f(z) = \dfrac{z^5 + 2}{z - 1}$. $\rightarrow$ poles at $1, \infty \in \hat{\mathbb{C}}$.

Clearly, $z = 1$. is a pole. We inspect ∞.

$$g(z) = f\left(\frac{1}{z}\right) = \frac{\frac{1}{z^5} + 2}{\frac{1}{z} - 1} = \frac{1 + 2z^5}{1 - z} \cdot \frac{1}{z^4} \quad \text{pole at } z = 0$$

$\Longrightarrow$ f has a pole at ∞.

Residue at ∞

$\operatorname{Res}(f, \infty) = ?$

Beware

$$\operatorname{Res}(f,\infty) \neq \operatorname{Res}(g,0).$$

Instead define

$$\boxed{\operatorname{Res}(f,\infty) := -\frac{1}{2\pi i}\int_{|z|=\rho} f\, dz}$$ where $\rho > R$.

By Homotopy Cauchy this does not depend on $\rho > R$

Question

clever computational device

Why do we care about the residue at ∞?

Homework Example

$$\int_{|z|=5} \frac{z^3}{(1-z)(2-z)(3-z)(4-z)}\, dz = -2\pi i \operatorname{Res}(\_\_, \infty).$$

This is better than computing 4 different residues.

Question How do we calculate the residue at ∞?

Answer

$$\operatorname{Res}(f, \infty) = - \operatorname*{Res}_{w=0} \left(g(w) \cdot \frac{1}{w^2} \right)$$

Proof Let ρ be sufficiently large. Then

$$z = \frac{1}{w}.$$

$$\operatorname{Res}(f, \infty) = -\frac{1}{2\pi i} \int_{|z|=\rho} f \, dz = \qquad dz = -\frac{dw}{w^2}.$$

$$= \frac{1}{2\pi i} \int_{|w| = 1/\rho} g \, \frac{-dw}{w^2} \quad \text{(change variables)}$$

(the change of orientation yields an extra sign).

$$= \operatorname*{Res}_{w=0} \left(g(w) \cdot \frac{-1}{w^2} \right)$$

using the usual residue theorem.

Residue Theorem for $\hat{\mathbb{C}}$

If f has isolated singularities only at $a_1, \dots, a_k \in \mathbb{C}$ and possibly at ∞ then

$$\sum_{a \in \hat{\mathbb{C}}} \operatorname{Res}(f, a) = 0.$$

Proof Let ρ be large enough, $\rho > |a_j|$ for all j.

$$\operatorname{Res}(f, \infty) = -\frac{1}{2\pi i} \int_{|z|=\rho} f \, dz \quad \text{(definition)}$$

$$= -\sum_j \operatorname{Res}(f, a_j) \quad \text{(usual residue theorem)}$$

$$\Longrightarrow \sum_{a \in \hat{\mathbb{C}}} \operatorname{Res}(f, a) = 0.$$

Remark* This generalizes correctly to other compact Riemann surfaces.

Example $f(z) = \dfrac{z^5+2}{z-1}$.

we saw that f has pole at $z=1$ and $z=\infty$.

$$\mathrm{Res}(f,1) \overset{\text{Method 1}}{=} \left.\frac{z^5+2}{(z-1)'}\right|_{z=1} = 3.$$

$$\mathrm{Res}(f,\infty) = \mathrm{Res}_{w=0}\left(g(w)\cdot\frac{-1}{w^2}\right)$$

$$z = 1/w \quad \Rightarrow g(w) = f\left(\frac{1}{w}\right) = \frac{1/w^5+2}{1/w - 1} = \frac{1+2w^5}{1-w}\cdot\frac{1}{w^4}.$$

Thus $\quad \mathrm{Res}(f,\infty) = \mathrm{Res}_{w=0}\; \dfrac{1+2w^5}{1-w}\cdot\dfrac{1}{w^4}\cdot\dfrac{-1}{w^2}$

$$= \mathrm{Coeff}_{w^5} - \frac{1+2w^5}{1-w}$$

$$= \mathrm{Coeff}_{w^5} - (1+2w^5)(1+w+w^2+w^3+w^4+w^5)$$

$$= -(1+2) = -3$$

This is consistent with the residue theorem on $\hat{\mathbb{C}}$.

Example (Lagrange)

Let $f(z) = \dfrac{P(z)}{Q(z)}$. Assume that

- $\deg P = p$, $\deg Q = q$, $p \leq q - 2$
- Q has simple roots $\alpha_1, \dots, \alpha_q$

f has poles at $\alpha_1 \dots \alpha_q$ and possibly at ∞.

- $\mathrm{Res}(f, \alpha_i) \overset{\text{Method 1}}{=} \dfrac{P(\alpha_i)}{Q'(\alpha_i)}$
- $\mathrm{Res}(f, \infty) = 0$ (next page).

Residue Theorem for $\hat{\mathbb{C}}$ $\Longrightarrow$ $\displaystyle\sum_{i=1}^{q} \frac{P(\alpha_i)}{Q'(\alpha_i)} = 0$

When $P(z) = z^p$, $Q(z) = \prod_{i=1}^{q} (z - \alpha_i)$, this gives

$$\sum_{i=1}^{q} \frac{\alpha_i^{\,p}}{\prod_{j \neq i} (\alpha_j - \alpha_i)} = 0 \qquad \forall\, p \leq q-2. \qquad \forall\, \alpha_1 \dots \alpha_q \text{ distinct}$$

Proof $\quad \operatorname{Res}\left(\frac{P}{Q}, \infty\right) = 0 \quad$ if $p \leq q-2$.

Write $P = a_0 z^p + \dots + a_p$, $a_0 \neq 0$

$Q = b_0 z^q + \dots + b_q$, $b_0 \neq 0$.

$$\operatorname{Res}\left(\frac{P}{Q}, \infty\right) = \operatorname{Res}_{w=0}\left(\frac{a_0 \frac{1}{w^p} + a_1 \frac{1}{w^{p-1}} + \dots + a_p}{b_0 \frac{1}{w^q} + b_1 \frac{1}{w^{q-1}} + \dots + b_q} \cdot \frac{-1}{w^2}\right)$$

$$= \operatorname{Res}_{w=0}\left(\frac{w^q}{w^p} \cdot \frac{a_0 + a_1 w + \dots + a_p w^p}{b_0 + b_1 w + \dots + b_q w^q} \cdot \frac{-1}{w^2}\right)$$

$$= -\operatorname{Res}_{w=0}\left(\underbrace{w^{q-p-2} \cdot \frac{a_0 + a_1 w + \dots + a_p w^p}{b_0 + b_1 w + \dots + b_q w^q}}_{\text{holomorphic near } 0 \text{ since } p+2 \leq q}\right)$$

$= 0$.

Example

$$I = \int_{|z|=10} \frac{z^8\, dz}{(z-1)(z-2)\cdots(z-9)} = \,?$$

Residue theorem:

residue thm for $\hat{\mathbb{C}}$

$$I = 2\pi i \sum_{k=1}^{9} \operatorname{Res}(f, k) = -2\pi i \operatorname{Res}(f, \infty) = 2\pi i.$$

Indeed,

$$\operatorname{Res}(f, \infty) = \operatorname{Res}\left(f\left(\frac{1}{z}\right)\cdot\frac{-1}{z^2}, 0\right) =$$

$$= \operatorname{Res}\left(\frac{1/z^8}{\left(\frac{1}{z}-1\right)\cdots\left(\frac{1}{z}-9\right)}\cdot\frac{-1}{z^2}, 0\right)$$

$$= \operatorname{Res}\left(\frac{-1/z}{(1-z)\cdots(1-9z)}, 0\right)$$

$$= -\frac{1}{(1-z)\cdots(1-9z)}\Big/_{z=0} = -1.$$

$$\Rightarrow \quad I = 2\pi i$$

Remark* (will not use)

Better to speak about residue of forms

$$f\,dz \xleftrightarrow{\text{versus}} f$$

Remark A change of coordinates near 0 is a biholomorphism

$$h : U \longrightarrow V, \quad h(0) = 0, \quad h'(0) \neq 0$$

where U, V are open neighborhoods of 0

Residues of functions are dependent of choice of coordinates

i.e.

$$\operatorname{Res}(f, 0) \neq \operatorname{Res}(f \circ h, 0).$$

Example $f(z) = \dfrac{1}{z}$, $h(z) = \lambda z$, $\lambda \neq 0$.

$$\operatorname{Res}(f, 0) = 1$$

$$\operatorname{Res}(f \circ h, 0) = \frac{1}{\lambda}$$

Residues of forms are coordinate-independent!

$$\operatorname*{Res}_{z=0}(f(z)\,dz) = \operatorname*{Res}_{w=0}(f(h(w))\,d\,h(w)) \qquad z = h(w)$$

$$= \operatorname*{Res}_{z=0}(f(h(w)).\,h'(w)\,dw).$$

Indeed $\displaystyle\int_\gamma f(z)\,dz = \int_{h^{-1}(\gamma)} f(h(w)).\,h'(w)\,dw$

Change of variables formula

This independence applies to the residue at ∞ as well:

change of variables $z = 1/w$

$$\operatorname*{Res}_{z=\infty}(f\,dz) = \operatorname*{Res}_{w=0}\left(g(w).\,d\left(\frac{1}{w}\right)\right)$$

$$= \operatorname*{Res}_{w=0}\left(g(w)\cdot\frac{-dw}{w^2}\right). \qquad g(w) = f\left(\frac{1}{w}\right)$$

This justifies the choice of sign in the definiton of the residue at ∞.