Math 220 A — Lecture 16

November 27, 2023

$\boxed{1.}$ **The Argument Principle** is a useful application of the **Residue Theorem.**

Order $f : U \longrightarrow \mathbb{C}$ meromorphic, $U \subseteq \mathbb{C}$, $a \in U$.

$$\operatorname{ord}(f, a) = \begin{cases} n, & a \text{ zero of order } n \\ -n, & a \text{ pole of order } n \\ 0, & \text{otherwise} \end{cases}$$

Remarks $\boxed{i}$ $\operatorname{ord}(f, a) = n \iff f = (z-a)^n g$

where g holomorphic near a, $g(a) \neq 0$

This follows by inspecting the Taylor / Laurent expansion.

$\boxed{ii}$ $\operatorname{ord}(fg, a) = \operatorname{ord}(f,a) + \operatorname{ord}(g,a)$

Indeed, let $\operatorname{ord}(f,a) = m$, $\operatorname{ord}(g,a) = n$.

Write $f = (z-a)^m F$, $g = (z-a)^n G$, $F(a), G(a) \neq 0$

$\Rightarrow fg = (z-a)^{m+n} FG$ with $FG(a) \neq 0$.

$\boxed{i}$
$\Rightarrow \operatorname{ord}(fg, a) = m+n = \operatorname{ord}(f,a) + \operatorname{ord}(g,a)$.

Question Find poles & residues of $\frac{f'}{f}$

Answer Poles of $\frac{f'}{f}$ come from zeros or poles of f.

Let a be a zero/pole with $\mathrm{ord}(f, a) = k$.

$\Rightarrow f = (z-a)^k g$, g holomorphic, $g(a) \neq 0$.

$$\Rightarrow \frac{f'}{f} = \frac{k(z-a)^{k-1} g + (z-a)^k g'}{(z-a)^k g} = \frac{k}{z-a} + \frac{g'}{g}$$

Since $g \neq 0$ near a $\Rightarrow \frac{g'}{g}$ holomorphic near a

$\Rightarrow \frac{f'}{f}$ has simple pole and

$$\boxed{\mathrm{Res}\left(\frac{f'}{f}, a\right) = \mathrm{ord}(f, a)} \quad (= k).$$

Argument Principle / Conway V. 3.4.

Theorem Given f meromorphic in $\mathcal{U}$, $\gamma \overset{\mathcal{U}}{\sim} 0$, avoiding the zeros and poles of f, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \sum_a n(\gamma, a)\,\mathrm{ord}(f, a)$$

This follows by the Residue Theorem & above discussion.

Remarks $\boxed{\text{i}}$ In practice, γ is a circle or a simple closed curve with $\mathrm{Int}\,\gamma \subseteq \mathcal{U}$. Then

$$n(\gamma, a) = \begin{cases} 1, & a \in \mathrm{Int}\,\gamma \\ 0, & a \in \mathrm{Ext}\,\gamma \end{cases}$$

Thus

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \#\,\text{Zeroes} - \#\,\text{Poles in } \mathrm{Int}\,\gamma.$$

(counted with multiplicity)

$\boxed{\text{ii}}$ We have $\displaystyle\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{dw}{w}$ for $w = f(z)$

$$= n(f\circ\gamma, 0) = \text{winding number.}$$

[iii] Why is it called "argument principle"?

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \frac{1}{2\pi i}\int_\gamma d\log f$$

$$= \frac{1}{2\pi i}\,\Delta \log f$$

$$= \frac{1}{2\pi \cancel{i}}\,\Delta\left(\underbrace{\cancel{\log|f|}}_{0} + \cancel{i}\,\mathrm{Arg} f\right)$$

$$= \frac{1}{2\pi}\,\Delta\,\mathrm{Arg} f$$

Conway V.3.6.

[iv] Enhanced version $g : U \to \mathbb{C}$ holomorphic

f meromorphic in U, $\gamma \overset{U}{\sim} 0$ avoiding the zeros

and poles of f,

$$\frac{1}{2\pi i}\int_\gamma g\,\frac{f'}{f}\,dz = \sum_a g(a)\cdot n(\gamma,a)\cdot \mathrm{ord}(f,a)$$

follows from Enhanced Residue Thm (PSet 6 #3)

If γ is simple closed, $\mathrm{Int}\,\gamma \subseteq \mathcal{U}$, then

$$\frac{1}{2\pi i}\int_\gamma g\,\frac{f'}{f}\,dz = \sum g(\text{zeros } f) - g(\text{poles of } f)$$

Int γ Int γ

appear with multiplicity

Proof We apply the Residue theorem.

We show $\mathrm{Res}\left(g\cdot\frac{f'}{f}, a\right) = g(a)\ \mathrm{ord}(f,a)$

Let $\mathrm{ord}(f,a) = k$. We know from page 2:

$$\frac{f'}{f} = \frac{k}{z-a} + F, \quad F, G \text{ holomorphic near } a$$

$$g = g(a) + (z-a)G \quad \text{(Taylor expansion)}$$

$$\Longrightarrow g\cdot\frac{f'}{f} = \left(\frac{k}{z-a} + F\right)\left(g(a) + (z-a)G\right)$$

$$= \frac{k\,g(a)}{z-a} + H \quad \text{where } H \text{ holomorphic near } a$$

$$\Longrightarrow \mathrm{Res}\left(g\cdot\frac{f'}{f}, a\right) = k\,g(a) = \mathrm{ord}(f,a)\cdot g(a).$$

2. Applications of the Argument Principle

- elliptic functions
- biholomorphisms
- Rouché's theorem

} next time

Elliptic functions

← see HNK 4

- studied by Abel, Jacobi, Weierstraß
- connected with arclength of ellipse

  elliptic integrals

  elliptic curves

- rich theory but we only say a few words here

  (more in Math 220 B)

Carl Gustav Jacob Jacobi (1804 – 1851)

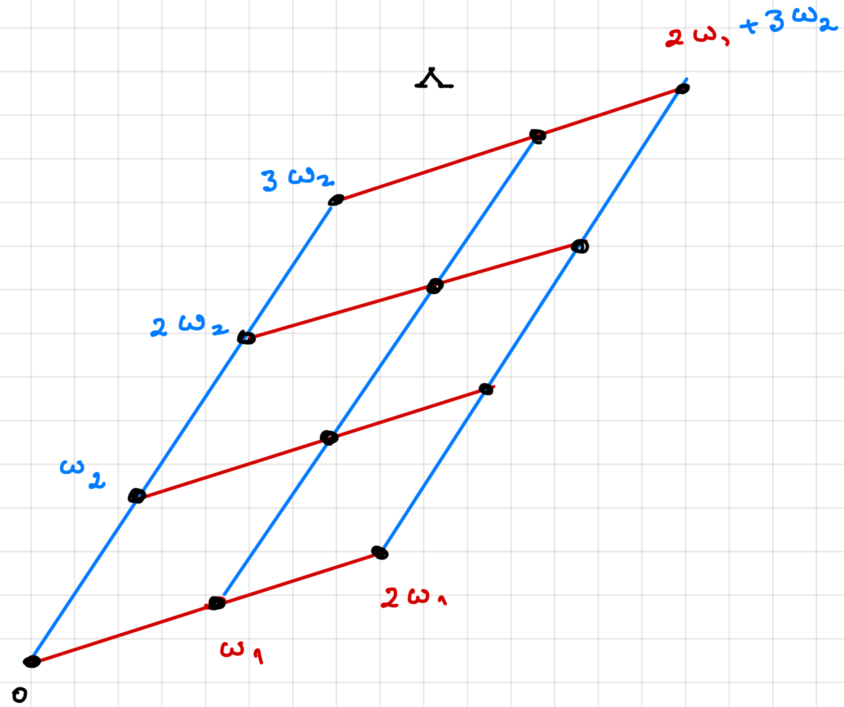
Jacobian, Jacobi symbol, Jacobi identity, symbol ∂

Karl Weierstraß (1815 – 1897)

Definition

Let $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$, $\dfrac{\omega_1}{\omega_2} \notin \mathbb{R}$. Define the lattice

$$\Lambda = \mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2 = \{ m\,\omega_1 + n\,\omega_2 \;:\; m,n \in \mathbb{Z} \}$$



Def An elliptic function f satisfies

[i] f meromorphic on $\mathbb{C}$

[ii] f periodic, $f(z) = f(z+\omega_1) = f(z+\omega_2)$

Note that in fact $\forall\, \lambda \in \Lambda$, $f(z) = f(z+\lambda)$ $\quad (*)$

Remark The best-known elliptic function is

Weierstraß

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right)$$

We will study this function in detail later in

226B

Remark f elliptic $\Rightarrow f'$ elliptic.

Indeed $f(z) = f(z+\lambda) \Rightarrow f'(z) = f'(z+\lambda)$ for $\lambda \in \Lambda$.

Basic Properties of Elliptic Functions

Note that Λ is a subgroup of $\mathbb{C}$.

Define

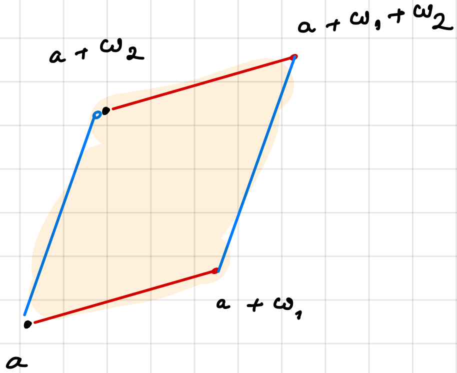
$$z \equiv w \mod \Lambda \iff z - w \in \Lambda.$$

$$z \equiv w \mod \Lambda \overset{(*)}{\Longrightarrow} f(z) = f(w).$$

Remark f is determined by values mod Λ

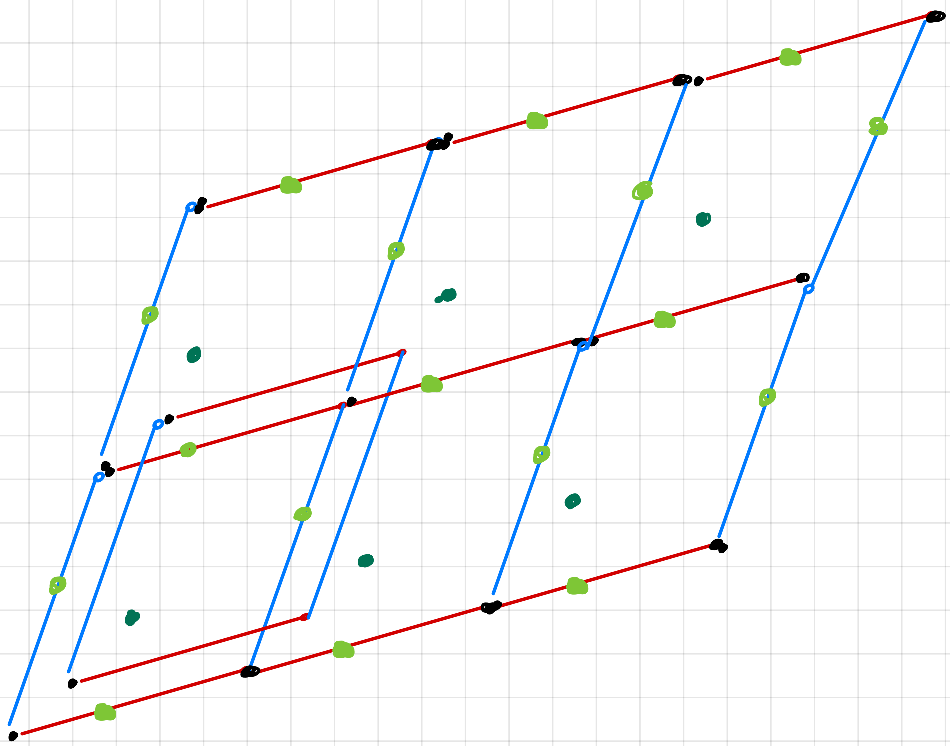
We will restrict f to a parallelogram.

$$P_a = \{ a + t_1 \omega_1 + t_2 \omega_2 : 0 \le t_1 \le 1,\ 0 \le t_2 \le 1 \}.$$



Each point in $\mathbb{C}$ is congruent to a point in P_a.

(see next picture)



Claim $\exists$ a such that ∂P_a contains no zeroes / poles. of f.

Proof Start with any a. Since P_a is compact & zeroes / poles are discrete $\Rightarrow \exists$ finitely many of them in P_a. A suitable translation would ensure ∂P_a avoids them.

Write $P = P_a$ where P is chosen as above.

Remark (HWK 4, # 7)

If f holomorphic in $\mathbb{C}$ $\Rightarrow$ $f|_P$ continuous

$\overset{P \text{ compact}}{\Longrightarrow}$ $f|_P$ bounded

$\overset{\text{periodic}}{\Longrightarrow}$ f bounded

$\overset{\text{Liouville}}{\Longrightarrow}$ f constant

Thus in general f will have poles.

Notation zeros in P : $\alpha_1 \dots \alpha_k$ (w/ multiplicity)

poles in P : $\beta_1 \dots \beta_\ell$ (w/ multiplicity)

Theorem [i] $k = \ell$: # zeroes (f) = # Poles (f) in P

[ii] $$\sum_{i=1}^{k} \alpha_i \equiv \sum_{i=1}^{k} \beta_i \mod \Lambda.$$

Remark Given $\alpha_1 \dots \alpha_k$, $\beta_1 \dots \beta_k$ with

$$\sum_i \alpha_i \equiv \sum_i \beta_i \mod \Lambda$$

there is an elliptic function with these zeroes/poles.

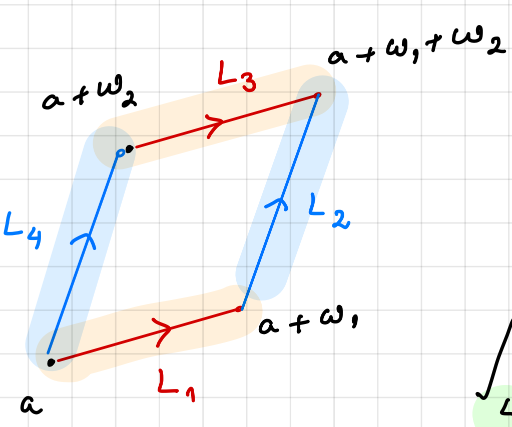
This is not obvious. $\rightsquigarrow$ Abel-Jacobi theory

Proof [L] By the Argument Principle

$$\frac{1}{2\pi i} \int_{\partial P} \frac{f'}{f} \, dz = \# \text{ Zeroes}(f) - \# \text{ Poles}(f) \text{ in } P.$$

We show $\int_{\partial P} \frac{f'}{f} \, dz = 0$. Let $\partial P = L_1 + L_2 + (-L_3) + (-L_4)$

We show $\int_{L_1} \frac{f'}{f} \, dz = \int_{L_3} \frac{f'}{f} \, dz$ & $\int_{L_2} \frac{f'}{f} \, dz = \int_{L_4} \frac{f'}{f} \, dz$.



Both claims follow by periodicity.

$$\int_{L_1} \frac{f'}{f} \, dz = \int_{L_1} \frac{f'}{f}(z + \omega_2) \, dz = \int_{L_3} \frac{f'}{f} \, dz$$

$$(L_3 = L_1 + \omega_2).$$

[4] We use the Enhanced Argument Principle $(g(z) = z)$.

$$\frac{1}{2\pi i}\int_{\partial P} z\,\frac{f'}{f}\,dz = \sum_{i=1}^{k}\alpha_i - \sum_{i=1}^{k}\beta_i .$$

We show $$\frac{1}{2\pi i}\left(\int_{L_1} z\,\frac{f'}{f}\,dz - \int_{L_3} z\,\frac{f'}{f}\,dz\right)\in\Lambda$$ and

$$\frac{1}{2\pi i}\left(\int_{L_2} z\,\frac{f'}{f}\,dz - \int_{L_4} z\,\frac{f'}{f}\,dz\right)\in\Lambda$$

This will complete the proof.

We only consider 1st expression. $L_3 = L_1 + \omega_2$

f periodic

$$\frac{1}{2\pi i}\left(\int_{L_1} z\,\frac{f'}{f}\,dz - \int_{L_3} z\,\frac{f'}{f}\,dz\right) = \frac{1}{2\pi i}\left(\int_{L_1} \cancel{z}\,\frac{f'}{f}\,dz - \int_{L_1}(\cancel{z}+\omega_2)\,\frac{f'}{f}\,dz\right)$$

$$= -\frac{1}{2\pi i}\,\omega_2\cdot\int_{L_1}\frac{f'}{f}\,dz$$

$w = f(z)$.

$$= -\left(\frac{1}{2\pi i}\int_{f(L_1)}\frac{dw}{w}\right)\cdot\omega_2$$

$$= -\underbrace{n\,(f(L_1),0)}_{\text{integer}}\,\omega_2 \in \Lambda .$$

Note that $f(L_1)$ is a loop (by periodicity). not containing 0.