Math 220 A — Lecture 17

November 29, 2023

Last time — Argument Principle

Theorem
- f meromorphic in U
- $g : U \to \mathbb{C}$ holomorphic
- $\gamma \overset{U}{\sim} 0$ avoiding the zeros and poles of f

Then

$$\frac{1}{2\pi i} \int_\gamma g \frac{f'}{f} \, dz = \sum_a g(a) \cdot n(\gamma, a) \cdot \operatorname{ord}(f, a)$$

Applications

I. elliptic functions (last time)

II. biholomorphisms

III. Rouché's theorem

1. Biholomorphisms

Recall $f: U \longrightarrow V$ is a biholomorphism if

[i] f bijective

[ii] f holomorphic

[iii] f^{-1} holomorphic

Remark The open mapping theorem shows that if $f: U \longrightarrow V$ satisfies [i] + [ii], then f^{-1} is continuous.

(Indeed, f open $\Rightarrow f^{-1}$ continuous).

More strongly, we will show that condition [iii] is automatic assuming [i] & [ii]. This is special to complex analysis.

The holomorphicity of f^{-1} is a local question near each point. It suffices to consider the following situation.

Let $f: U \longrightarrow \mathbb{C}$ holomorphic & injective. Let $\overline{\Delta} \subseteq U$,

and set $W = f(\Delta) =$ open. Note

$f: \Delta \longrightarrow W$ bijective

Conway V. 3.7

Proposition The following integral formula for the inverse function holds

$$f^{-1}(q) = \frac{1}{2\pi i} \int_{\partial\Delta} z \cdot \frac{f'(z)}{f(z)-q}\, dz \quad \forall\, q \in W$$

In particular $f^{-1}: W \longrightarrow \Delta$ is holomorphic.

Remark For $q \in f(\Delta)$, $f-q$ has no zeros on $\partial\Delta$

since $f|_{\overline{\Delta}}$ is injective by assumption.

Proof Apply the Enhanced Argument Principle to

$f - q$ and $g(z) = z$. Since f injective, $\exists!\ p \in \Delta$ with

$f(p) = q$. $\Longrightarrow f^{-1}(q) = p$. But

$$\frac{1}{2\pi i} \int_{\partial\Delta} z \, \frac{f'(z)}{f(z)-q}\, dz = g(p) = p = f^{-1}(q).$$

To show f^{-1} is holomorphic, recall Conway IV. 2.3, HWK3.

Key statement $\varphi : U \times \{\gamma\} \longrightarrow \mathbb{C}$

- φ continuous
- $z \longrightarrow \varphi(z, w)$ holomorphic $\forall\, w \in \{\gamma\}$.

Then $g(z) = \int_\gamma \varphi(z, w)\, dw$ holomorphic.

Apply this to $\varphi : \underset{\substack{\cup \\ g}}{V} \times \underset{\substack{\cup \\ z}}{\partial\Delta} \longrightarrow \mathbb{C}$

$$\varphi(g, z) = z \cdot \frac{f'(z)}{f(z) - g}$$. continuous &

holomorphic in g $\forall\, z \in \partial\Delta$. Then

$$f^{-1}(g) = \frac{1}{2\pi i} \int_{\partial\Delta} \varphi(g, z)\, dz = \text{holomorphic in } g.$$

$\Downarrow$

Corollary

A bijective holomorphic function has holomorphic inverse.

$\boxed{2.}$ Rouché's theorem (Conway V. 3. 8)

Idea $f, g : U \longrightarrow \mathbb{C}$ holomorphic

$$f = \underbrace{g}_{\text{dominant term}} + \text{lower order terms}$$

$\Longrightarrow$ # zeroes (f) = # zeroes (g). (w/ multiplicity).

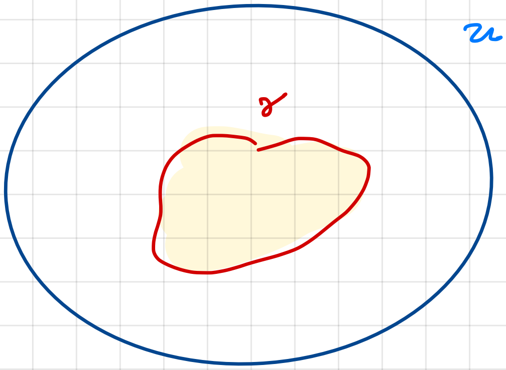
Moral:

We can ignore the lower order terms.

Setup: $\gamma \subseteq U$ simple closed curve, $\operatorname{Int} \gamma \subseteq U$.

e.g. $\gamma = \partial \Delta$, $\overline{\Delta} \subseteq U$.

$\rightsquigarrow$ typical case



Then

$$n(\gamma, a) = \begin{cases} 1, & a \in \operatorname{Int} \gamma \\ 0, & a \in \operatorname{Ext} \gamma \end{cases}$$

Theorem $f, g : U \longrightarrow \mathbb{C}$ holomorphic, γ as above.

If $|f-g| < |g|$ on γ $\Rightarrow$

$$\# \text{ zeroes}(f) = \# \text{ zeroes}(g). \text{ in } \operatorname{Int}(\gamma).$$

(w/ multiplicity)

Note that $f \neq 0$ & $g \neq 0$ on γ.

Remark Conway's version is more general but this level of generality is not needed in practice.

Conway's version

- f, g meromorphic
- $|f-g| < |f| + |g|$ on γ

$$\Rightarrow \# \text{ zeroes}(f) - \# \text{ Poles}(f) = \# \text{ zeros}(g) - \# \text{ Poles}(g)$$

in $\operatorname{Int} \gamma$.

Eugene Rouché

(1832 – 1910)

Proof (see Conway for a different proof)

Let $h_t = g + t(f-g)$, $0 \le t \le 1$.

Claim $t \longrightarrow \#\,\text{zeroes}(h_t)$ is continuous in t. (in Int γ)

This implies $\#\,\text{zeroes}(h_t) = $ constant.

Since $h_0 = g$, $h_1 = f \Longrightarrow \#\,\text{zeroes}(f) = \#\,\text{zeroes}(g)$.

Proof of the claim We use the Argument Principle

$$\#\,\text{zeroes}(h_t) = \frac{1}{2\pi i} \int_\gamma \frac{h_t'(z)}{h_t(z)}\, dz.$$

Note $|h_t| = |g + t(f-g)| \ge |g| - |t|\,|f-g|$

$\ge |g| - |f-g| > 0$ on γ.

Set $\psi(t,z) = \dfrac{h_t'(z)}{h_t(z)}$; $[0,1] \times \{\gamma\} \longrightarrow \mathbb{C}$.

Note ψ is continuous.

Key Fact $\quad \psi : [0,1] \times \{\gamma\} \longrightarrow \mathbb{C}$ continuous

$$\Rightarrow \quad \Phi(t) = \int_\gamma \psi(t, z)\, dz \text{ is continuous in } t.$$

Quick proof: Since $[0,1] \times \{\gamma\}$ is compact, ψ is uniformly continuous.

Fix $\varepsilon > 0$. Then $\exists\, \delta > 0$ with

$$|t - t'| < \delta \Rightarrow |\psi(t, z) - \psi(t', z)| \leq \varepsilon.$$

$$\Rightarrow |\phi(t) - \phi(t')| = \left| \int_\gamma \psi(t, z) - \psi(t', z)\, dz \right|$$

$$\leq \varepsilon \cdot \text{length}(\gamma)$$

$\Rightarrow \phi$ continuous.

Examples

dominant term

$\boxed{1}$ $f = z^5 + 24z^3 + 2z^2 + 3z + 1$

Question

How many roots does f have in $|z| < 1$?

Answer

Let $g = 24z^3$. and $\gamma = \{|z| = 1\}$.

We verify $|f - g| < |g|$. when $|z| = 1$.

Note $|g| = 24\,|z|^3 = 24.$

triangle inequality

$$|f - g| = |z^5 + 2z^2 + 3z + 1| \leq |z|^5 + 2|z|^2 + 3|z| + 1$$

$$= 1 + 2 + 3 + 1 = 6.$$

$\Longrightarrow$ $|f - g| < |g|$ on γ

$\Longrightarrow$ $\#\,\text{Zeroes}(f) = \#\,\text{Zeroes}(g) = 3$ in $\{|z| < 1\}$.

Example [ii] Fundamental Theorem of Algebra

Let $f = z^n + a_1 z^{n-1} + \dots + a_n$

Then $g = z^n$ = dominant term when $|z|$ large.

Indeed, $f - g = a_1 z^{n-1} + \dots + a_n$.

When $|z| = R$,

$$|f-g| \leq |a_1| R^{n-1} + \dots + |a_n| < R^n = |z|^n = |g|.$$

This happens for R large as $\lim_{R \to \infty} \dfrac{R^n}{|a_1| R^{n-1} + \dots + |a_n|} = \infty$.

By Rouché:

$$\# \text{ Zeroes }(f) = \# \text{ Zeroes }(g) = n \quad \text{in } \Delta(0,R).\ \forall R \gg 0$$

$\Rightarrow$ # Zeroes $(f) = n$. in $\mathbb{C}$

This gives another proof of the Fundamental Theorem of Algebra.

We can also use Rouché for nonpolymomial functions.

Example $\boxed{III}$

$$f(z) = e^z - 5z^3 + 1. \quad , \quad \gamma = \{ |z| = 1 \}.$$

How many zeroes does f have in the unit disc ?

Dominant term $g(z) = -5z^3$.

Indeed $|g| = 5$ for $|z| = 1$.

$$|f-g| = |e^z + 1| \leq |e^z| + 1 = e^{\operatorname{Re} z} + 1$$

$$\leq e^{|z|} + 1 = e + 1 < 5 = |g|$$

$\Rightarrow$ # zeroes (f) = # zeroes $(g) = 3$ in $\Delta(0,1)$.

Example $\boxed{IV}$

Let $h : \mathcal{U} \longrightarrow \mathbb{C}$, $\overline{\Delta}(0,1) \subseteq \mathcal{U}$, $|h(z)| < 1$, $|z| = 1$.

Then h has one fixed point in $\Delta(0,1)$.

Proof We show $h(z) = z \iff h(z) - z = 0$ has a unique solution in $\Delta(0,1)$.

Let $f(z) = h(z) - z$, $g(z) = -z$, $\gamma = \{|z| = 1\}$.

Then

$$|f - g| = |h| < 1 = |g| \quad \text{on } \gamma$$

$\Rightarrow$ # zeros (f) = # zeros $(g) = 1$. $\Rightarrow$

$\Rightarrow$ h has a unique fixed point in $\Delta(0,1)$.

Remark Hurwitz's theorem (next week) will be another application of Rouché!