Math 220 A – Lecture 18

December 4, 2023

$\boxed{I}$ Sequences of holomorphic functions (Conway $\overline{\underline{VII}}$).

Outline
- notions of convergence
- Weierstraß' theorem
- Hurwitz's theorem $\Longleftarrow$ Rouché

Types of convergence

Question What is the correct notion of convergence for holomorphic functions?

$f_n : \mathcal{U} \longrightarrow \mathbb{C}$, $f : \mathcal{U} \longrightarrow \mathbb{C}$ be any functions.

Math 140B $\boxed{i}$ pointwise convergence

$$f_n \longrightarrow f \quad \text{iff} \quad \forall\, x \in \mathcal{U}, \; f_n(x) \longrightarrow f(x).$$

$\boxed{ii}$ uniform convergence

$$f_n \rightrightarrows f \quad \text{iff} \quad \sup_{\mathcal{U}} |f_n - f| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Issues

$\boxed{\text{i}}$ Pointwise convergence does not preserve continuity

$$f_n : [0,1] \to \mathbb{R}, \quad f_n(x) = x^n \longrightarrow f(x) = \begin{cases} 0 & , x \neq 1 \\ 1 & , x = 1 \end{cases}.$$

$\boxed{\text{ii}}$ Uniform convergence is better, & preserves continuity

$$f_n \rightrightarrows f \ \& \ f_n \text{ cont} \implies f \text{ continuous} \quad (\text{Math } 140\text{B})$$

However, the notion is too strong. Natural examples don't pass the bar:

$$f_n(z) = \frac{z}{n}, \quad f(z) = 0, \quad f_n \not\rightrightarrows f \text{ on } \mathbb{C}.$$

$\boxed{\text{iii}}$ In real analysis, both notions are not well-behaved under differentiation

$$f_n(x) = \frac{\sin nx}{\sqrt{n}} \longrightarrow 0, \quad f_n'(0) = \sqrt{n} \longrightarrow \infty$$

We consider a slightly weaker notion.

Better [a] uniform convergence on compact sets

[b] local uniform convergence

[a] Notation : $f_n \xrightarrow{c} f$ or $f_n \overset{c}{\rightrightarrows} f$

Definition $\forall\ K \subseteq \mathcal{U}$ compact, $\sup_K |f_n - f| \longrightarrow 0$ as $n \to \infty$.

[b] Notation $f_n \overset{l.u.}{\rightrightarrows} f$

Definition : $\forall\ x \in \mathcal{U}$ $\underbrace{\exists\ \Delta(x, r_x) \subseteq \mathcal{U}}_{\text{local}}$ with $\underbrace{f_n \rightrightarrows f}_{\text{uniform converg.}}$ in $\Delta(x, r_x)$.

Claim [a] = [b].

Thus $f_n \xrightarrow{c} f$, $f_n \overset{c}{\rightrightarrows} f$, $f_n \overset{l.u.}{\rightrightarrows} f$ mean the same thing.

Proof [a] $\Rightarrow$ [b]. If [a] holds for all K, take

$K = \overline{\Delta}(x, r_x) \subseteq \mathcal{U}$., K compact. This choice of K yields [b].

$\boxed{b} \Longrightarrow \boxed{a}$. Let $f_n \overset{l.u.}{\rightrightarrows} f$. Take K compact in $\mathcal{U}$.

For $x \in K$, $\exists\, \Delta(x, r_x)$ with $f_n \rightrightarrows f$ in $\Delta(x, r_x)$.

Since $K \subseteq \bigcup_{x \in K} \Delta(x, r_x) \Longrightarrow K \subseteq \bigcup_{j=1}^{N} \Delta(x_j, r_{x_j})$

by compactness. Since

$$\sup_K |f_n - f| \leq \max_{1 \leq j \leq N} \left(\sup_{\Delta(x_j, r_{x_j})} |f_n - f| \right) \longrightarrow 0$$

$\Longrightarrow$ $f_n \rightrightarrows f$ in K $\Longrightarrow f_n \overset{c}{\rightrightarrows} f$.

<u>Example</u> $f_n = \frac{z}{n}$, $f = 0$, $f_n \overset{c}{\rightrightarrows} f$ in $\mathbb{C}$.

Indeed, $\sup_K |f_n - f| = \sup_{z \in K} \left| \frac{z}{n} \right| \leq \frac{M}{n} \longrightarrow 0$.

so $f_n \overset{c}{\rightrightarrows} f$. This was the example disallowed before.

Remark (Continuity & Math 140B).

f_n continuous & $f_n \rightrightarrows f$ then f continuous

f_n continuous & $f_n \overset{l.u}{\rightrightarrows} f$ then f continuous.

(because continuity is a local concept).

Important Convention

$\mathcal{C}(\mathcal{U})$ = continuous functions in $\mathcal{U}$

$\mathcal{O}(\mathcal{U})$ = holomorphic functions in $\mathcal{U}$

We will always consider local uniform convergence. for both $\mathcal{O}(\mathcal{U})$ and $\mathcal{C}(\mathcal{U})$.

2. Weierstraß' Theorem

Let $f_n : U \longrightarrow \mathbb{C}$ holomorphic, $f_n \overset{l.u.}{\rightrightarrows} f$. Then

$\boxed{1}$ f holomorphic

$\boxed{2}$ $f_n^{(k)} \overset{l.u.}{\rightrightarrows} f^{(k)} \quad \forall\, k \geq 0$

This is very different than in real analysis.

Remark $\boxed{1}$ $\mathcal{O}(U) \hookrightarrow \mathcal{C}(U)$ "closed." under local uniform limits.

$\boxed{1c}$ integration is not an issue

If $f_n \overset{l.u.}{\rightrightarrows} f$ then $f_n \rightrightarrows f$ since $\{\gamma\}$ compact, hence

$$\int_\gamma f_n \, dz \longrightarrow \int_\gamma f \, dz.$$ via the basic estimate.

Proof [i] Let $\bar{R} \subseteq \mathcal{U}$ closed rectangle, ∂R = compact.

Since $f_n \overset{\text{l.u.}}{\rightrightarrows} f \Rightarrow \int_{\partial R} f_n \, dz \longrightarrow \int_{\partial R} f \, dz$

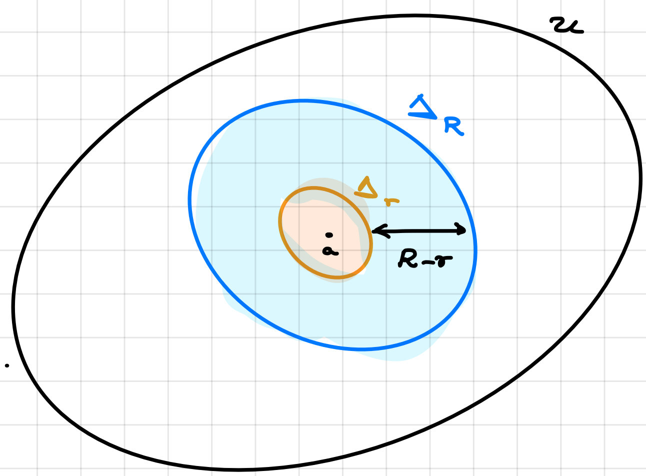
Since f_n holomorphic $\Rightarrow \int_{\partial R} f_n \, dz = 0$. (Cauchy). Take $n \to \infty$.

$\Rightarrow \int_{\partial R} f \, dz = 0 \Rightarrow f$ admits a primitive F in any disc in $\mathcal{U}$.

$\Rightarrow f = F'$ = holomorphic in any disc $\Rightarrow f$ holomorphic.

[ii] By induction, suffices to show

$$f_n' \overset{\text{l.u.}}{\rightrightarrows} f' \text{ in } \mathcal{U}$$



$|w - z| \geq |w| - |z|$

$\geq R - r$.

Let $a \in \mathcal{U}$, $\underset{\overline{\Delta_r}}{\overline{\Delta}(a, r)} \subseteq \underset{\overline{\Delta_R}}{\overline{\Delta}(a, R)} \subseteq \mathcal{U}$, $r < R$.

Suffices $f_n' \rightrightarrows f'$ in $\overline{\Delta}_r$.

We use CIF for $z \in \overline{\Delta}_r$

$$\left| f_n'(z) - f'(z) \right| = \left| \frac{1}{2\pi i} \int_{\partial \Delta_R} \frac{f_n(w) - f(w)}{(w-z)^2} \, dw \right|$$

$$\leq \frac{1}{\cancel{2\pi}} \cdot \sup_{\partial \Delta_R} |f_n - f| \cdot \frac{1}{(R-r)^2} \cdot \cancel{2\pi} R$$

Thus $\displaystyle \sup_{\overline{\Delta}_r} |f_n' - f'| \leq \frac{R}{(R-r)^2} \cdot \sup_{\partial \Delta_R} |f_n - f| \longrightarrow 0$ as $n \to \infty$.

$\Rightarrow f_n' \overset{l.u.}{\rightrightarrows} f'$.

$\boxed{3.}$ <u>Series</u> $f_n : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic. Assume

$(*)$ $\forall$ $K \subseteq \mathcal{U}$ compact $\exists$ $M_n(K)$, $|f_n| \leq M_n(K)$.

over K. & $\displaystyle\sum_{n=1}^{\infty} M_n(K) < \infty$.

M-test

$\Longrightarrow$ $f = \displaystyle\sum_{n=1}^{\infty} f_n$ converges absolutely & uniformly on every K.

Weierstraß Thm $\Longrightarrow$ f holomorphic & $f' = \displaystyle\sum_{n=1}^{\infty} f_n'$

<u>Remark</u> We have seen a particular case of this for power series. (Lecture 2).

<u>Definition</u> If $\forall K \subseteq \mathcal{U}$ compact, we have

$$\sum_n \sup_K |f_n| < \infty ,$$

then we say $\displaystyle\sum_n f_n$ converges normally.

Remark Normal convergence is only defined for series not for sequences.

Remark If $\sum f_n$ converges normally $\Rightarrow$ $\sum f_n$ converges absolutely & locally uniformly.

Remark If $\sum_{n=1}^{\infty} f_n$ converges normally to f, then all rearrangements $\sum_{n=1}^{\infty} f_{\sigma(n)}$ also converge normally to f. , $\sigma : \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}_{>0}$

Proof Rudin 3.55 (Math 140A) shows

$\sum a_n$ converges absolutely $\Rightarrow$ all rearrangements converge to the same sum.

Apply this to $a_n = \sup_K |f_n|$. Then $\sum f_n$ converges normally $\Rightarrow$ $\sum a_n$ converges absolutely $\Rightarrow$ $\sum a_{\sigma(n)}$ converges $\Rightarrow$ $\sum f_{\sigma(n)}$ converges normally.

Apply this again to $a_n = f_n(x)$, $x \in U$. Then $\sum f_n$ converges normally $\Rightarrow$ $\sum a_n$ converges absolutely to $f(x)$ & thus all rearrangements $\sum a_{\sigma(n)} = \sum f_{\sigma(n)}(x)$ equal $f(x)$.

Example (ζ - function)

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ gives a holomorphic function in $\operatorname{Re} s > 1$.

Take $f_n(s) = \frac{1}{n^s}$. holomorphic in s.

Take $K \subseteq \{\operatorname{Re} s > 1\}$. Since $\operatorname{Re} : K \longrightarrow \mathbb{R}$ is continuous,

it achieves a minimum on K $\Rightarrow$ $\operatorname{Re} z \geq \alpha$ $\forall z \in K$, $\alpha > 1$.

$$|f_n| = \left|\frac{1}{n^s}\right| = \left|\frac{1}{n^x} \cdot \cancel{\frac{1}{n^{iy}}}^{1}\right| = \frac{1}{n^x} \cdot 1. \leq \underbrace{\frac{1}{n^\alpha}}_{M_n}. \quad \text{where } s = x + iy$$

$\sum_{n=1}^{\infty} M_n < \infty$. by real analysis $\Longrightarrow$ $\sum_{n=1}^{\infty} f_n$ holomorphic. in s

$\Rightarrow \zeta$ holomorphic in s, $\operatorname{Re} s > 1$.

Remarks $\boxed{i}$ We have seen $\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n} B_{2n}}{2\,(2n)!}$ (HWK6)

$\boxed{ii}$ This can be extended holomorphically to $s \neq 1$

(requires work).