Math 220A – Lecture 19

December 6, 2023

1. Hurwitz' Theorem

Let $f_n : \mathcal{U} \to \mathbb{C}$ holomorphic, $f_n \overset{l.u.}{\rightrightarrows} f$, $\overline{V} \subseteq \mathcal{U}$ compact

If $f|_{\partial V}$ has no zeros, $\exists N$ such that

$$\underset{\overline{V}}{\#\text{Zeros}}(f) = \underset{\overline{V}}{\#\text{Zeros}}(f_n) \text{ for } n \geq N.$$

w/ multiplicity

Proof

[1] Most useful case (Conway VII. 2. 5)

$V = \Delta(a, R)$

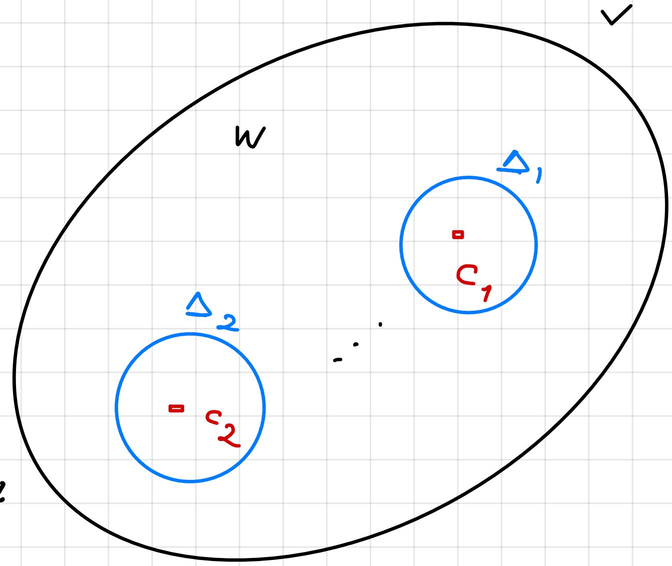
Since $f|_{\partial V}$ has no zeroes $\Rightarrow \varepsilon = \min_{\partial V} |f| > 0.$

Since $f_n \rightrightarrows f$ over $\partial V \Rightarrow \exists N$ s.t. over ∂V $\forall n \geq N.$

$\sup_{\partial V} |f_n - f| < \varepsilon \leq \min_{\partial V} |f| \Rightarrow |f_n - f| < |f|$ over $\partial V.$

$\Rightarrow$ Rouché: $\#\text{zeroes}(f) = \#\text{zeroes}(f_n)$ in $\overline{V}.$

[ii] General case



$\overline{V}$ compact $\Rightarrow$ f has finitely many zeroes $c_1 \dots c_k$ in $\overline{V}$.

Surround c_j by small disjoint discs Δ_j, $W = V \setminus \bigcup_j \overline{\Delta_j}$

$\Rightarrow$ f has no zeros in $\overline{W}$ $\Rightarrow \exists N$ such that $\forall n \geq N$, f_n has no zeros in $\overline{W}$. (Indeed, let $\varepsilon = \min_{\overline{W}} |f| > 0$. Then $\exists N$ such that $|f_n - f| < \varepsilon$ on $\overline{W}$ for $n \geq N$. This implies $f_n \neq 0$ on $\overline{W}$ for $n \geq N$.).

Then,

$$\#\,\text{Zeroes}_{\overline{V}}(f) = \sum_{j=1}^{k} \#\,\text{Zeroes}_{\overline{\Delta_j}}(f) =$$

for n large by 1^{st} case applied to f_n on $\overline{\Delta_j}$.

$$= \sum_{j=1}^{k} \#\,\text{Zeroes}_{\overline{\Delta_j}}(f_n) =$$

using f_n has no zeroes in $\overline{W}$

$$= \#\,\text{Zeroes}_{\overline{V}}(f_n) \text{ for } n \geq N.$$

open connected

Corollary A $f_n \overset{l.u.}{\rightrightarrows} f$, f_n holomorphic in $\mathcal{U}$,

If f_n is zero free $\forall n \implies f$ zero - free or $f \equiv 0$.

$\rightsquigarrow$ Conway VII. 2.6.

Proof Indeed if $f \not\equiv 0$, let a be chosen so that

$f(a) = 0$. Let $V = \bar{\Delta}(a,r)$, $f|_{\partial V}$ has no zeroes.

(Argue by contradiction using zeros cannot accumulate.)

Hurwitz

$$\implies \underbrace{\#\,\text{zeroes}\,(f_n)_{\bar{V}}}_{0.} = \#\,\text{Zeroes}\,(f)_{\bar{V}} \geq 1. \quad \forall\, n \geq N.$$

assumption.

a is a zero. contradiction.

$\implies f$ is zero - free

Example

$\mathcal{U} = \mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$

- $f_n(z) = z$, $f(z) = z$, $f_n \overset{l.u.}{\rightrightarrows} f$, f zero free.
- $f_n(z) = \frac{z}{n}$, $f(z) = 0$, $f_n \overset{l.u.}{\rightrightarrows} f$, $f \equiv 0$.

Thus both situations can occur.

Remark

The corresponding statement fails in real analysis.

Take $f_n : \mathbb{R} \longrightarrow \mathbb{R}$, $f_n(x) = x^2 + \frac{1}{n}$

$f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$.

Note $f_n \rightrightarrows f$, f_n has no zeros, but f does.

Corollary B

$f_n \overset{l.u.}{\rightrightarrows} f$, f_n holomorphic in $\mathcal{U}$,

If f_n are injective $\forall n$ $\Rightarrow$ f injective or f constant.

Proof. Assume f not injective, $f(a) = f(b)$, $a \neq b$.

$\tilde{f}_n = f_n - f_n(a)$.

$\tilde{f} = f - f(a)$.

Since $\left.\begin{array}{l} f_n(a) \longrightarrow f(a) \\ f_n \overset{l.u}{\rightrightarrows} f \end{array}\right\} \Rightarrow \tilde{f}_n \overset{l.u.}{\rightrightarrows} \tilde{f}$.

f_n injective $\Rightarrow$ $\tilde{f}_n$ zero free on $\tilde{\mathcal{U}} = \mathcal{U} \setminus \{a\}$.

$\overset{\text{Corollary A}}{\Rightarrow}$ $\tilde{f}$ is zero free on $\tilde{\mathcal{U}}$ or $\tilde{f} \equiv 0$. on $\tilde{\mathcal{U}}$

Note that $\tilde{f}(b) = f(b) - f(a) = 0$ $\Rightarrow$ $\tilde{f}$ is not zero-free in $\tilde{\mathcal{U}}$. Thus $\tilde{f} \equiv 0$ in $\tilde{\mathcal{U}}$ $\Rightarrow$ f constant.

Adolf Hurwitz (1859 – 1919)

2. Summary of topics for Math 220A

(1) Holomorphic functions. Harmonic functions. The Cauchy-Riemann equations.

(2) Conformal maps. Fractional linear transformations. Cayley transform.

(3) Existence of primitives. Logarithm. Winding numbers.

(4) Cauchy's theorem and Cauchy's integral formula, different versions. Cauchy's estimates.

(5) Taylor and Laurent series.

(6) Zeroes of holomorphic functions, identity principle, open mapping theorem, maximum modulus principle, Liouville's theorem.

(7) Types of singularities. Removable singularities theorem. Poles. Essential singularities. Meromorphic functions. Residues. Cassorati-Weierstraß.

(8) The residue theorem. Residues at infinity. Applications to real analysis.

(9) The argument principle. Rouché's theorem.

(10) Sequences and series of holomorphic functions. Weierstraß convergence theorem. Hurwitz's theorem.

Final Exam Tuesday, Dec 12, 11:30 – 2:30

Office Hours – Monday, Dec 11, 1 – 2:30

Review – Homeworks, lectures

– Practice Problems online

– Old Final & Solutions

3. Preview of Math 220B

Part I : Sequences / Series / Products

(1) infinite products of holomorphic functions

Weierstraß Problem

(2) sequences & series of meromorphic functions

Mittag-Leffler Problem

(3) sequences of hol functions, Montel families

Part II : Geometric aspects / Conformal maps

(4) Schwarz Lemma, automorphisms of $\Delta, \mathfrak{H}^+, \Delta^*, \dots$

(5) Riemann mapping theorem

Part III : Further topics (depending on time)

(6) Runge's theorem

(7) Schwarz Reflection

Three Motivating Questions for Part I

$f \not\equiv 0$ entire has countably many zeroes that do not accumulate in $\mathbb{C}$.

Weierstraß Problem

Given a sequence of distinct $\{a_n\}$, $a_n \to \infty$ and positive integers $\{m_n\}$, is there an entire function with zeroes only at $\{a_n\}$ with order exactly $\{m_n\}$?

Mittag-Leffler Problem

Take $\{a_n\}$ as above.

We can always find a meromorphic function f in $\mathbb{C}$ with poles only at a_n. e.g. take g solving Weierstraß at $\{a_n\}$ and set $f = 1/g$.

Mittag - Leffler asks if we can furthermore prescribe the Laurent principal parts.

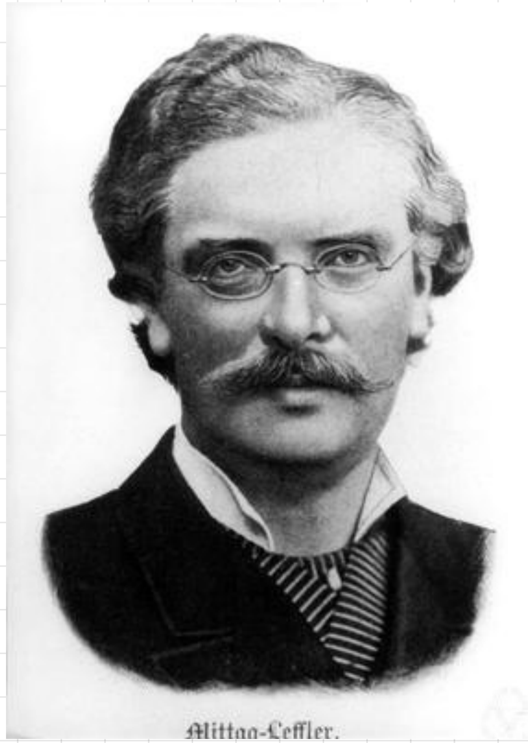
Given $\{a_n\}$ distinct, $a_n \to \infty$, and polynomials $p_n\left(\frac{1}{z-a_n}\right)$ without constant terms, is there a meromorphic function in $\mathbb{C}$ with poles only at a_n and Laurent expansion

$$f = p_n\left(\frac{1}{z-a_n}\right) + \dots \quad \text{near } a_n.$$

Weierstraß - Poincaré Problem

Is any meromorphic function a quotient of two holomorphic functions?

These questions are further connected with other areas of mathematics. (number theory, complex geometry, algebraic geometry)



Karl Weierstraß

1815 – 1897

Gösta Mittag-Leffler

1846 – 1927

Tools — sequences, series, products of holomorphic & meromorphic functions.

This quarter [I] sequences
[II] series
of holomorphic functions.

Next quarter

[I] Weierstraß requires infinite products of holomorphic functions.

Intuitively, this makes sense. We could try to solve Weierstraß by setting $f(z) = \prod_{n=1}^{\infty} (z - a_n)$ but convergence is an issue

[II] Mittag-Leffler requires infinite sums of meromorphic functions.

Part II investigates the following questions

Question A Given $U, V \subseteq \mathbb{C}$ are U, V biholomorphic?

Remark This has implications in topology & differential geometry. In particular U, V are

homeomorphic, diffeomorphic

Very Important Theorem (Riemann Mapping Theorem)

Given $U, V \neq \mathbb{C}$, U, V simply connected $\Rightarrow$ U, V are biholomorphic.

Question B

Given $U, V \subseteq \mathbb{C}$ biholomorphic can we construct

$\boxed{i}$ one biholomorphism $U \longrightarrow V$ explicitly?

$\boxed{ii}$ all biholomorphism $U \longrightarrow V$ explicitly?

Question C

What are all biholomorphisms $f : U \longrightarrow U$?

Connections w/ Lie theory, hyperbolic geometry.

Part III concerns approximation by polynomials & Runge's theorem.

Recall Any continuous function $f : [a,b] \to \mathbb{R}$ can be approximated by polynomials $p_n \rightrightarrows f$.

Question Can this be extended to holomorphic functions?

The answer is quite subtle & is provided by Runge's theorem.