Math 220 A - Lecture 2

Oct 4, 2023

Conway III.1, III.2.

[1] Power series & Analytic functions

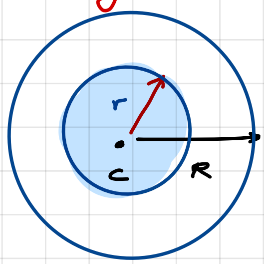
$c \in \mathbb{C}$, $a_n \in \mathbb{C}$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n \qquad (*)$$

Def / Theorem $\exists R$ $0 \le R \le \infty$ such that

[i] if $|z-c| < R \implies (*)$ converges.

If $0 \le r < R \implies (*)$ converges absolutely & uniformly in $\Delta(c, r)$.



[ii] if $|z-c| > R \implies (*)$ diverges.

Furthermore $R^{-1} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. R = radius of convergence.

Definition $f : \mathcal{U} \longrightarrow \mathbb{C}$ is analytic if $\forall z_0 \in \mathcal{U}$ $\exists R > 0$

such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{in} \quad \Delta(z_0, R) \subseteq \mathcal{U}.$$

Proof WLOG $c = 0$. , else work $z^{new} = z - c$.

$\sum_{n=0}^{\infty} a_n z^n$. We set $R^{-1} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Let $|z| < r$

$\boxed{i}$. Let $r < \rho < R$. $\Rightarrow \limsup \sqrt[n]{|a_n|} = \frac{1}{R} < \frac{1}{\rho} \Rightarrow$

$$\Rightarrow \sqrt[n]{|a_n|} < \frac{1}{\rho} \quad \text{if } n \geq N.$$

$$\Rightarrow |a_n| < \frac{1}{\rho^n} \quad \text{if } n \geq N$$

$$\Rightarrow \underbrace{|a_n z^n|}_{f_n(z).} < \underbrace{\left(\frac{r}{\rho}\right)^n}_{M_n} \quad \text{if } n \geq N.$$

Weierstraß M-test

$|f_n| \leq M_n$, $\sum_n M_n < \infty$ then $\sum_n f_n$ converges absolutely & uniformly

$\Rightarrow \sum_n a_n z^n$ converges absolutely & uniformly in $\Delta(0, r)$.

1c If $|z| > \rho > R \Rightarrow \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R} > \frac{1}{\rho}$

$\Rightarrow \sqrt[n]{|a_n|} > \frac{1}{\rho}$ for infinitely many n's

$\Rightarrow |a_n| > \frac{1}{\rho^n}$ for infinitely many n's

$\Rightarrow |a_n z^n| > \underbrace{\left(\frac{|z|}{\rho}\right)}_{>1}^n$ for infinitely many n's

$\Rightarrow a_n z^n \not\to 0$

$\Rightarrow \sum_n a_n z^n$ diverges.

Differentiation

Recall that if $f_n \longrightarrow f$ it doesn't follow $f_n' \longrightarrow f'$ in general. However, for power series we have

Theorem (Rudin 8.1).

If $\sum_{n\geq 0} a_n (z-c)^n$ has radius of convergence R, then $\sum_{n\geq 1} n\, a_n (z-c)^{n-1}$ has radius of convergence R as well.

Furthermore, if

$$f(z) = \sum_{n\geq 0} a_n (z-c)^n \quad \text{in} \quad \Delta(c,R)$$

$$\Longrightarrow f'(z) = \sum_{n\geq 1} n\, a_n (z-c)^{n-1} \quad \text{in} \quad \Delta(c,R).$$

Corollary $f^{(k)}(z) = \sum_{n\geq k} a_n\, n(n-1)\cdots(n-k+1)\, (z-c)^{n-k}$

$$\overset{z=c}{\Longrightarrow} \quad f^{(k)}(c) = a_k\, k! \quad \Rightarrow a_k = \frac{f^{(k)}(c)}{k!}$$

$$\Rightarrow f(z) = \sum_{n\geq 0} \frac{f^{(n)}(c)}{n!} (z-c)^n \quad \text{if } f \text{ is analytic in } \Delta(c,R).$$

Remark If f is analytic $\overset{\text{Thm}}{\Longrightarrow}$ f is holomorphic.

Proof WLOG $c=0$. $\rightsquigarrow \sum_{n=0}^{\infty} a_n z^n$

Convergence of the series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ is equivalent to convergence of $\sum_{n=0}^{\infty} n a_n z^n = z\left(\sum_{n=1}^{\infty} n a_n z^{n-1}\right)$. The radius of convergence is $R'^{-1} = \limsup_{n\to\infty} \sqrt[n]{|n a_n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = R^{-1} \Rightarrow$

$\Rightarrow R' = R$ using $\sqrt[n]{n} \to 1$.

For the second statement, let $\alpha \in \Delta(0,R)$. We show $f'(\alpha) = g(\alpha)$ where $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. Let

$$S_N(z) = \sum_{n=0}^{N} a_n z^n, \quad R_N = \sum_{n=N+1}^{\infty} a_n z^n$$

Know $S_N \to f$, $S_N' \to g$.

Fix $\varepsilon > 0$. Wish to find $\delta_\varepsilon > 0$ with

$$\left| \frac{f(z) - f(\alpha)}{z - \alpha} - g(\alpha) \right| < \varepsilon \quad \text{for } z \in \Delta(\alpha, \delta_\varepsilon)$$

Let $|\alpha| < \rho < R$.

For $z \in \Delta(0,\rho)$ we have

$$(*) = \left| \frac{f(z) - f(\alpha)}{z-\alpha} - g(\alpha) \right| \leq \left| \frac{S_N(z) - S_N(\alpha)}{z-\alpha} \overset{\text{I.}}{-} S_N'(\alpha) \right|$$

$$+ \left| S_N'(\alpha) \overset{\text{II}}{-} g(\alpha) \right|$$

$$+ \left| \frac{R_N(z) \overset{\text{III.}}{-} R_N(\alpha)}{z-\alpha} \right| \overset{?}{<} \varepsilon.$$

We estimate each of these terms. Term $\overline{\text{III}}$

$$\left| \frac{R_N(z) - R_N(\alpha)}{z-\alpha} \right| \leq \sum_{n=N+1}^{\infty} |a_n| \left| \frac{z^n - \alpha^n}{z-\alpha} \right|$$

$$\leq \sum_{n=N+1}^{\infty} |a_n| \left(|z|^{n-1} + \dots + |\alpha|^{n-1} \right)$$

$$\leq \sum_{n=N+1}^{\infty} |a_n| n \rho^{n-1} < \varepsilon/3$$

if $N \geq N_1$.

for some N_1. This is due to the absolute convergence of

$$\sum_{n=0}^{\infty} n a_n \rho^{n-1} \quad (\text{since } \rho < R).$$

Term II Since $s_N' \longrightarrow g$, we find N_2 with

$$|s_N'(\alpha) - g(\alpha)| < \varepsilon/3 \quad \text{for } n \geq N_2.$$

Fix $N \geq N_1$, $N \geq N_2$. For this N, find δ such that

Term I: $$\left| \frac{s_N(z) - s_N(\alpha)}{z - \alpha} - s_N'(\alpha) \right| < \varepsilon/3 \quad \text{if } z \in \Delta(\alpha, \delta).$$

Conclusion

$$(*) \leq \text{Term I} + \text{Term II} + \text{Term III} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

if $z \in \Delta(\alpha, \delta) \cap \Delta(0, \rho)$. QED.

Examples : exp, cos, sin

$$\boxed{1}\quad e^{z} := \underbrace{1 + z + \frac{z^2}{2} + \dots + \frac{z^n}{n!} + \dots}_{f(z)}, \quad R = \infty. \quad \text{Indeed,}$$

$$R = \limsup_{n\to\infty} \sqrt[n]{n!} \geq \limsup_{n\to\infty} \sqrt[n]{\left(\frac{n}{2}\right)^{n/2}} = \limsup_{n\to\infty} \sqrt{\frac{n}{2}} = \infty. \quad \text{Differentiate}$$

$$f'(z) = 0 + 1 + z + \dots + \frac{z^{n-1}}{(n-1)!} + \dots = f(z)$$

$$\implies (e^z)' = e^z.$$

In a similar way, $(e^{z+c})' = e^{z+c}$.

This implies $e^{z+c} = e^z \cdot e^c$

Indeed, let $y = \dfrac{e^{z+c}}{e^z}$ and note

$$y' = \frac{(e^{z+c})' e^z - e^{z+c}(e^z)'}{e^{2z}} = \frac{e^{z+c}\cdot e^z - e^{z+c}\cdot e^z}{e^{2z}} = 0$$

$\Rightarrow y$ constant; $y(0) = e^c \Rightarrow y = e^c \Rightarrow e^{z+c} = e^z \cdot e^c$.

[ii] Define

$$\cos z := \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$\sin z := \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\Longrightarrow \quad (\sin z)' = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \cos z.$$

$$\sin^2 z + \cos^2 z = 1.$$

Beware ! $\sin z$, $\cos z$ are not bounded functions as $z \in \mathbb{C}$

$$\cos in\pi = \frac{e^{-n\pi} + e^{n\pi}}{2} \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \pm\infty.$$

[iii] z^n can be defined for all $n \in \mathbb{Z}$. if $z \neq 0$.

II. Logarithm

Remark

$e^{2\pi i} = 1 \implies$ exponential is not invertible.

$$\log 1 = 0, \pm 2\pi i, \pm 4\pi i, \dots, \pm 2n\pi i$$

$\rightsquigarrow$ how should we pick?

Question Define $\log z$. ?

Remark Issues also arise with $\sqrt[n]{z}$ and z^{α}.

These are related to the logarithm.

$$\sqrt[n]{z} \longleftrightarrow z^{\alpha} \quad \text{for } \alpha = \frac{1}{n}$$

Define $\quad z^{\alpha} := \exp(\alpha \log z)$

Def A **logarithm** $\ell : U \longrightarrow \mathbb{C}$ is a continuous function with $e^{\ell(z)} = z \quad \forall z \in U$.

Naturally, for this to make sense, we need $U \subseteq \mathbb{C} \setminus \{0\}$.

Question Does ℓ exist? Is it unique?

Sometimes — No

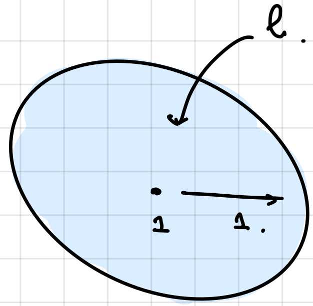
Remark If ℓ, ℓ' are two logarithms then

$$e^{\ell} = e^{\ell'} \Rightarrow e^{\ell - \ell'} = 1 \Rightarrow \ell - \ell' = 2\pi i n, \; n \in \mathbb{Z}.$$

Any two logarithms differ by $2\pi i n$, $n \in \mathbb{Z}$.

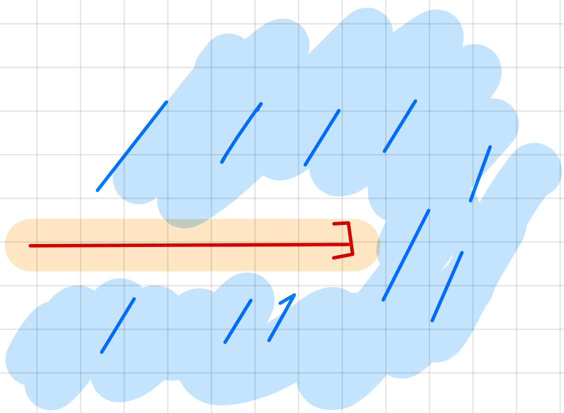
Example A $U = \Delta(1,1)$,

$$\ell(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n$$



This is a logarithm in U (homework)

Example B $\quad U = \mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\le 0}$ slit plane



$z = re^{i\theta}$. Set

$$\operatorname{Log} z = \log r + i\theta$$

$\theta \in (-\pi, \pi) \Rightarrow e^{\operatorname{Log} z} = z.$

(Principal branch of logarithm).

principal branch

Notice the capital letter in Log

BEWARE $\quad \operatorname{Log}(zw) \neq \operatorname{Log} z + \operatorname{Log} w$

This holds if $\operatorname{Re} z > 0$, $\operatorname{Re} w > 0$.

Remark The two examples above give the same answer in $\Delta(1,1)$.

Indeed the two logarithms $\ell(z)$ and $\operatorname{Log} z$ differ by $2\pi i n$ $\Rightarrow \operatorname{Log} z - \ell(z) = 2\pi i n$. Set $z = 1$

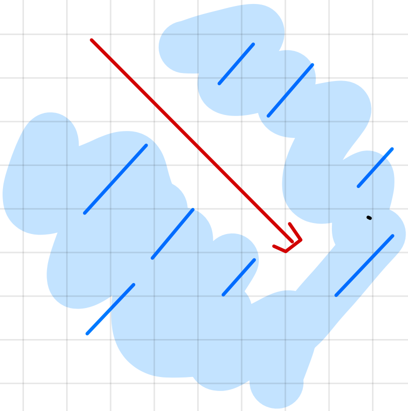
$$\Rightarrow \underbrace{\operatorname{Log} 1}_{0} - \underbrace{\ell(1)}_{0} = 2\pi i n \Rightarrow n = 0$$

$$\Rightarrow \operatorname{Log} z = \ell(z) \text{ in } \Delta(1,1).$$

Example $\operatorname{Log}(1-i) = \log\sqrt{2} + i\left(-\frac{\pi}{4}\right).$

principal branch

Example C Other branches of log by different cuts



$$U = \mathbb{C} \setminus \mathbb{R}_{\geq 0}\, e^{i\alpha}.$$

$$z = re^{i\theta}, \quad \theta \in (\alpha, \alpha + 2\pi).$$

$$\operatorname{Log}_\alpha z = \log r + i\theta.$$

Remark $\boxed{a}$ $U = \mathbb{C} \setminus \{0\} \Rightarrow$ impossible to define logarithm in U. why?

$\boxed{b}$ $U \subseteq \mathbb{C} \setminus \{0\}$ simply connected

$\Rightarrow$ we can define logarithm (later).

Examples A – C are simply connected.

Remark $z^{\alpha} = \exp(\alpha \, \ell(z))$ is multi-valued

values differ by $\exp(\alpha \cdot 2\pi i \cdot n)$, $n \in \mathbb{Z}$

Example Principal value is defined for $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

using principal branch of $\ell(z)$.

For instance, principal value

$$(1-i)^i = \exp(i \cdot \operatorname{Log}(1-i))$$

$$= \exp\left(i \cdot \left(\log\sqrt{2} - i\frac{\pi}{4}\right)\right)$$

$$= \exp\left(i \log\sqrt{2} + \frac{\pi}{4}\right)$$

$$= e^{\pi/4}\left(\cos\log\sqrt{2} + i \sin\log\sqrt{2}\right).$$