Math 220 A - Lecture 3

October 9, 2023

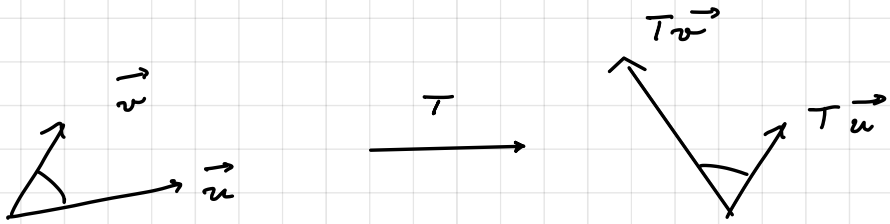
I. Conformal maps

Def $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\mathbb{R}$-linear, invertible

[i] T is orientation preserving if $\det T > 0$.

[ii] T is angle preserving if for any vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$

$$\measuredangle(\vec{u}, \vec{v}) = \measuredangle(T\vec{u}, T\vec{v}).$$



Remark $T = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is both orientation & angle preserving (unless $a = b = 0$).

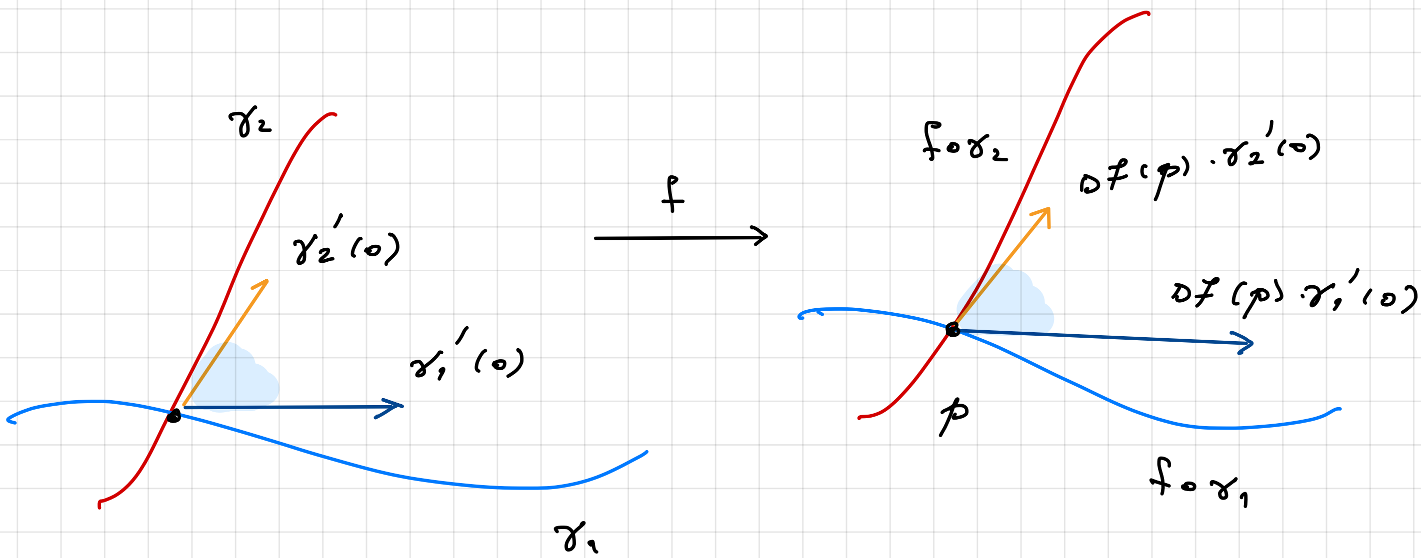
- $\det T = a^2 + b^2 > 0$ if $(a, b) \neq (0,0)$
- angles are preserved since $Tz = \alpha z = re^{i\theta} z$, $\alpha = re^{i\theta}$

is composition of rotation & dilation.

Remark f holomorphic $\Rightarrow$ either $f'(z) = 0$ or else

$Df(z)$ is orientation & angle preserving.

$\Rightarrow$ "f preserves angles" / f "conformal" if $f'(z) \neq 0$ $\forall z$.

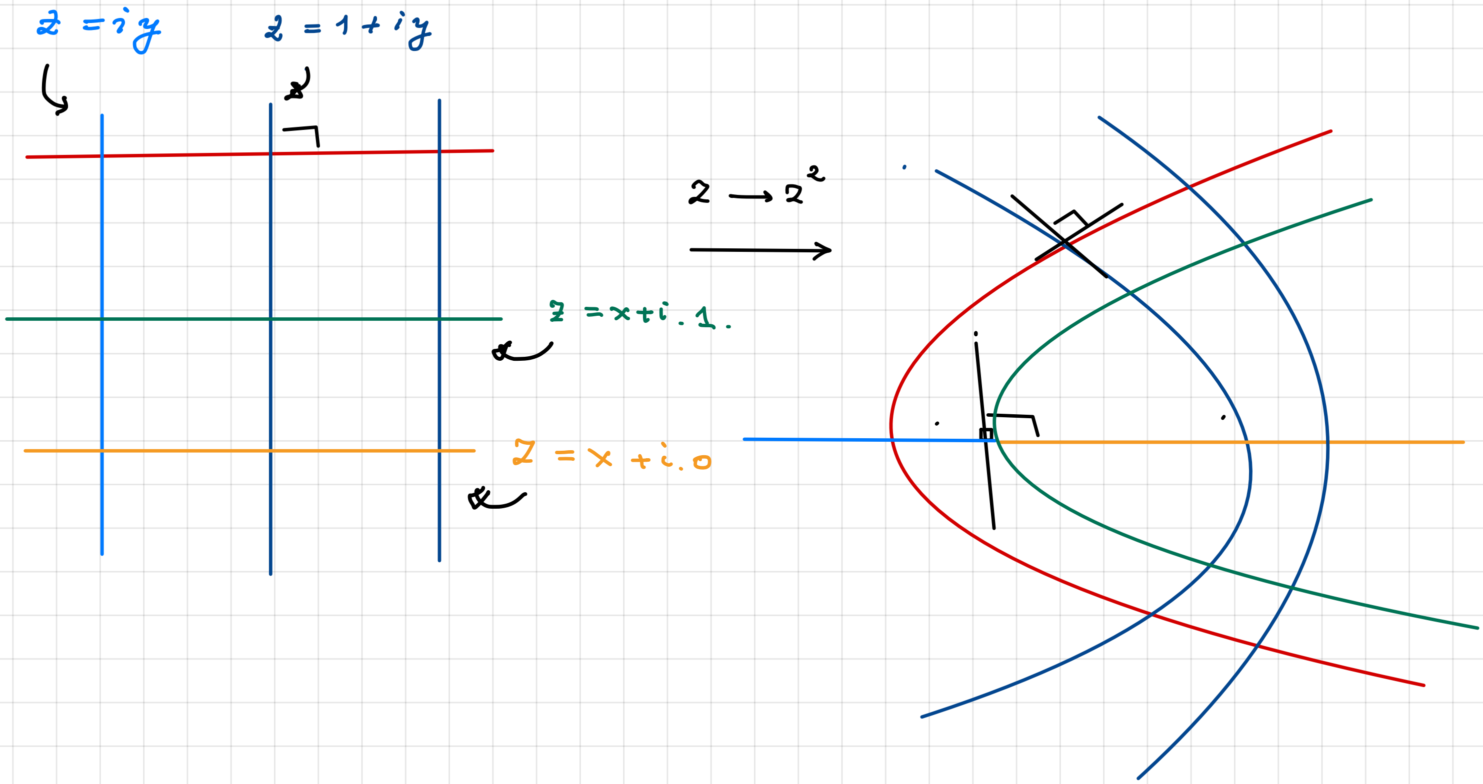


Def $f : U \longrightarrow \mathbb{C}$ holomorphic is conformal if

$$\forall z \in U, \; f'(z) \neq 0$$

Some other definitions exist in the literature (requiring f to be injective for instance).

Example $\quad f(z) = z^2$



$z = x + i.1 \implies z^2 = \underbrace{x^2 - 1}_{u} + \underbrace{2x}_{v}\, i$

$\implies u = \frac{v^2}{4} - 1 \quad$ parabola (green)

$z = iy \implies$ half line $\mathbb{R}_{\leq 0}$, $z = x \implies$ half line $\mathbb{R}_{\geq 0}$

$z = 1 + iy \implies z^2 = \underbrace{1 - y^2}_{u} + \underbrace{2y}_{v}\, i$

$\implies u = 1 - \frac{v^2}{4} \quad$ parabola (dark blue)

<u>Check</u> : Angles are preserved in these examples

A better piece of terminology is:

Remark Given $U, V \subseteq \mathbb{C}$, a biholomorphic map $f : U \longrightarrow V$ is

$\boxed{i}$ f bijective, holomorphic

$\boxed{ii}$ $g = f^{-1} : V \longrightarrow U$ holomorphic.

If $f(p) = q \implies f \circ g(z) = z$

$$\implies g'(q) = \frac{1}{f'(p)}, \quad f'(p) \neq 0.$$

Thus biholomorphic $\implies$ conformal

Important Question

Given $U, V \subseteq \mathbb{C}$, are they biholomorphic?

II. Möbius Transforms

Today we study a class of transformations which are important for geometric arguments.

Möbius transformations (MT)

Fractional linear transformations (FLT)

Linear fractional transformations (LFT)

August Ferdinand Möbius (1790 – 1868)

Möbius strip, Möbius inversion, Möbius transform

Möbius published important work in astronomy.

Definition $\hat{\mathbb{C}} = \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ Riemann sphere

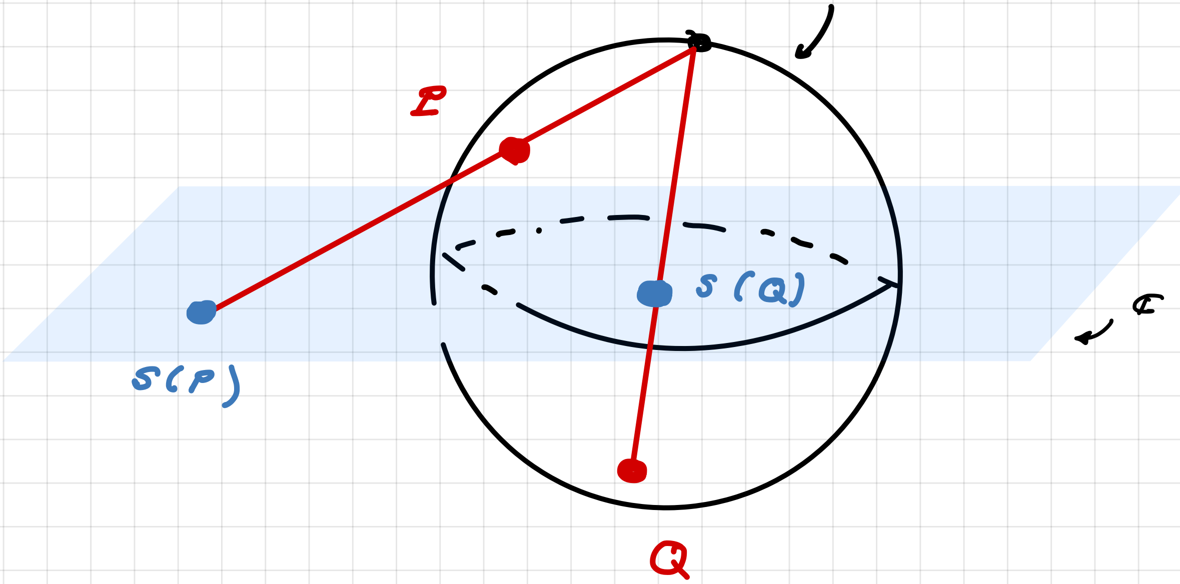
Topology on $\hat{\mathbb{C}}$
- usual open sets in $\mathbb{C}$,
- open sets containing ∞.

$\infty \in \mathcal{U} \subseteq \hat{\mathbb{C}}$ open if
- $\forall\, u \in \mathcal{U},\ u \neq \infty\ \ \exists\ \Delta(u, R) \subseteq \mathcal{U}$
- for $u = \infty$, $\exists\, R$ with

$$\{z : |z| > R\} \subseteq \mathcal{U}$$

$\rightsquigarrow$ neighborhood of ∞.

$S^2 \subseteq \mathbb{R}^3$ unit sphere



$\mathbb{C} \cup \{\infty\}$.

stereographic projection

Definition Möbius transformations MT.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad h_A : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}, \quad z \longrightarrow \frac{az+b}{cz+d}$$

invertible matrix

using $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$

This is a "biholomorphism" of $\hat{\mathbb{C}}$.

h_A is a homeomorphism of $\hat{\mathbb{C}}$.

Remark

$\boxed{i}$ $A = I \implies h_A = \mathbb{1}$.

$\boxed{ii}$ $A = \lambda B \iff h_A = h_B$ for $\lambda \neq 0$.

$\boxed{iii}$ $h_{AB} = h_A \circ h_B$ If $B = A^{-1} \implies h_{A^{-1}} = h_A^{-1}$.

Möbius transforms $\longleftrightarrow$ $PGL_2 = GL_2 / \text{center}$

"projective linear group"

Most famous example — Cayley transform

$$C(z) = \frac{z-i}{z+i}, \quad C^{-1}(w) = i \cdot \frac{1+w}{1-w}$$

Notation:

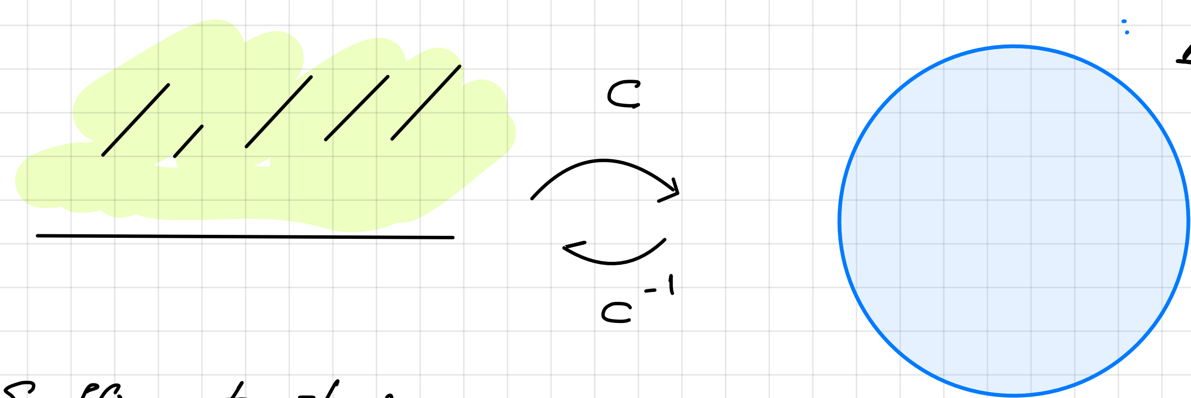
$\Delta = \Delta(0,1) = \text{unit disc}$

$\mathfrak{H}^+ = \{z : \operatorname{Im} z > 0\} = \text{half-plane}$

Very Important — C is a biholomorphism

$$C : \mathfrak{H}^+ \longrightarrow \Delta$$

$\mathfrak{H}^+$

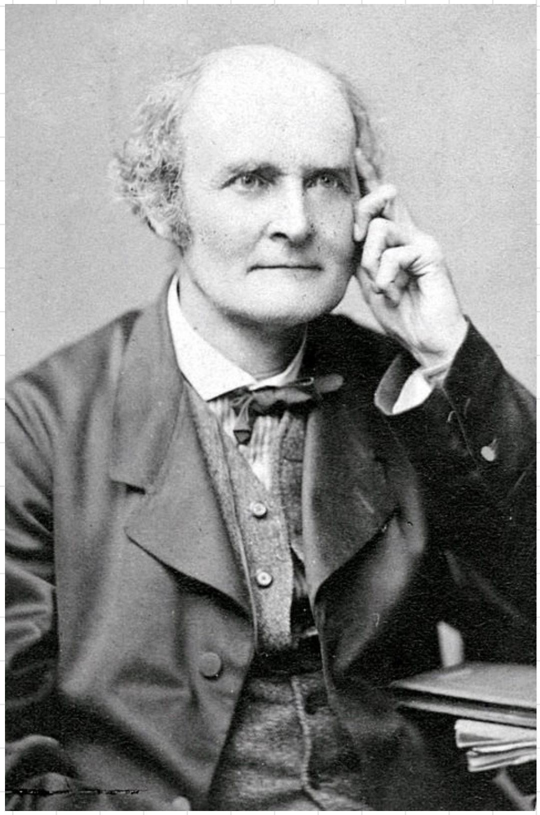


Proof

Suffices to show

$z \in \mathfrak{H}^+ \iff C(z) \in \Delta$. Write $z = x + iy$

$$\begin{array}{ccc} z \in \mathfrak{H}^+ & \iff & C(z) \in \Delta \\ \Updownarrow & & \Updownarrow \\ y > 0 & & |z-i| < |z+i| \\ & & \Updownarrow \\ & \iff & x^2 + (y-1)^2 < x^2 + (y+1)^2 \end{array}$$



Arthur Cayley (1821 - 1895)

- worked in algebraic geometry, Group theory
- Cayley - Hamilton theorem
- modern definition of a group

Remark

$$\frac{az+b}{cz+d} = \frac{bc-ad}{c^2}\,\frac{1}{z+\frac{d}{c}} + \frac{a}{c}$$

$c=0$: $$\frac{az+b}{d} = \frac{a}{d}\cdot z + \frac{b}{d}.$$

Types of Mobius transforms

$\boxed{\text{I}}$ translation $\quad Tz = z + \lambda$

$\boxed{\text{II}}$ rotations $\quad Rz = e^{i\theta}\cdot z$

$\boxed{\text{III}}$ dilations $\quad Dz = m\,z, \; m \in \mathbb{R}.$

$\boxed{\text{IV}}$ inversion $\quad Sz = \frac{1}{z}.$

Lemma All Möbius transforms are compositions of

$\boxed{\text{I}} - \boxed{\text{IV}}$

Generalized circles in $\hat{\mathbb{C}}$

$\boxed{\text{i}}$ circles in $\mathbb{C}$

$\boxed{\text{ii}}$ line $L \cup \{\infty\}$ = "circle in $\hat{\mathbb{C}}$ through ∞."

Main theorems about Mobius transforms

Theorem A Any Möbius transform maps generalized circles to generalized circles.

Theorem B PGL_2 acts triply transitively on $\hat{\mathbb{C}}$.

Given (z_1, z_2, z_3), (z_1', z_2', z_3') triples of distinct elts in $\hat{\mathbb{C}}$, $\exists!\, h$ with $h(z_i) = z_i'$

Proof of Thm A Suffices to consider the cases

[i] translation $z \longrightarrow z+\lambda$ clear

[ii] rotation $z \longrightarrow e^{i\alpha} z$ clear

[iii] dilation $z \longrightarrow m z$ clear

[IV] inversion $z \longrightarrow 1/z$.

Claim A generalized circle is given by

$$(*) \quad A z\bar{z} + Bz + C\bar{z} + D = 0.$$ where $A, D \in \mathbb{R}$.

and B, C are conjugates.

Proof A circle in $\mathbb{C}$ is given by

$$|z - z_0| = r \iff (z - z_0)\cdot(\bar{z} - \bar{z}_0) = r^2$$

$$\iff z\bar{z} - \bar{z}_0 z - z_0 \bar{z} + (z_0 \bar{z}_0 - r^2) = 0$$

$\Rightarrow$ $(*)$ for $A = 1$, $D = z_0\bar{z}_0 - r^2$, $B = -\bar{z}_0$, $C = -z_0$.

Conversely, if $A \neq 0$, $(*)$ can be brought into this form.

(why?)

When $A = 0$: $\underbrace{Bz + C\bar{z}}_{\text{linear}} + D = 0 \iff$ line.

Proof $\boxed{IV}$ preserves generalized circles.

$$A z\bar{z} + Bz + C\bar{z} + D = 0.$$

Let $w = \dfrac{1}{z} \Rightarrow A \cdot \dfrac{1}{w\bar{w}} + \dfrac{B}{w} + \dfrac{C}{\bar{w}} + D = 0$

$$\Rightarrow A + B\bar{w} + Cw + Dw\bar{w} = 0.$$

$$\Rightarrow \text{generalized circle.} \Rightarrow \text{Thm A.}$$

In the case of lines $L \cup \infty$, 0 and ∞ correspond under $\boxed{IV}$.

Proof of Thm B **Uniqueness** Assume $\exists\, h, h'$

$$z_1 \underset{h'}{\xrightarrow{h}} z_1'$$

$$z_2 \underset{h'}{\xrightarrow{h}} z_2'$$

$$z_3 \underset{h'}{\xrightarrow{h}} z_3'$$

Let $T = h'^{-1} \circ h \Rightarrow T(z_i) = z_i$,

$$\Leftrightarrow \frac{az+b}{cz+d} = z \text{ has 3 roots } z_1, z_2, z_3$$

$$\Leftrightarrow az+b = cz^2+dz \text{ has 3 roots}$$

$$\Rightarrow a = d,\ b = c \Rightarrow T = \mathbb{1} \Rightarrow h = h'!$$

Existence

Suffices: $\exists\, h$ with

$$
\begin{tikzcd}
z_1 \arrow[r, "h"] \arrow[rr, "T"'] & 0 & z_1' \arrow[l, "h'"'] \\
z_2 \arrow[r, "h"] \arrow[rr, "T"'] & 1 & z_2' \arrow[l, "h'"'] \\
z_3 \arrow[r, "h"] \arrow[rr, "T"'] & \infty. & z_3' \arrow[l, "h'"']
\end{tikzcd}
$$

$$h(z_1) = 0$$
$$h(z_2) = 1$$
$$h(z_3) = \infty.$$

If (z_1', z_2', z_3') is another triple, find h' with

$$h'(z_1') = 0\,, \quad h'(z_2) = 1, \quad h'(z_3) = \infty.$$

Define $T = h'^{-1} \circ h \implies T(z_i) = z_i'$ as needed.

To deal with (z_1, z_2, z_3) and $(0, 1, \infty)$.

__(Modified) Cross ratio__ If $z_1, z_2, z_3 \neq \infty$,

$$h(z) = \frac{z - z_1}{z - z_3} \Big/ \frac{z_2 - z_1}{z_2 - z_3}$$

This is sometimes denoted $[z : z_1 : z_2 : z_3]$.

Check $h(z_1) = 0$

$h(z_2) = 1$

$h(z_3) = \infty.$

There are 3 remaining case $z_1 = \infty$, $z_2 = \infty$ or $z_3 = \infty$.

For example, when $z_1 = \infty$. the above expression is

$$h(z) = \frac{z_2 - z_3}{z - z_3}, \quad h(z_1) = 0, \quad h(z_2) = 1, \quad h(z_3) = \infty.$$

The other cases are similar.

Remark Conway's cross ratio sends z_1, z_2, z_3 to $1, 0, \infty$.

We will not need the notion later, so the difference will not concern us.

Next Integration & Cauchy theory.