Math 220 A - Lecture 4

October 11, 2023

I. Cauchy theory & Integration (Conway IV)

The theory of integration is crucial to complex analysis. Many important results have as starting point Cauchy's integral formula.

§1. Complex integration

[a] $\mathcal{U} \subseteq \mathbb{C}$ open & connected

$\gamma : [a,b] \longrightarrow \mathcal{U}$ C^1- path

[i] length $(\gamma) = \int_a^b |\gamma'(t)|\, dt.$

[ii] C^1-reparametrization $\hat{\gamma} : [\hat{a}, \hat{b}] \longrightarrow \mathcal{U}$

$$\hat{\gamma} = \gamma \circ \Phi \,, \quad \underbrace{\Phi : [\hat{a}, \hat{b}] \longrightarrow [a,b]}_{\text{class } C^1}$$

Orientation preserving: $\phi' > 0$.

[b] A piecewise C^1-path

$$\gamma = \gamma_1 + \dots + \gamma_n\,, \quad \gamma_i \text{ of class } \mathcal{C}^1.$$

if $\exists$ $a = a_0 < a_1 < \dots < a_n = b$

$$\gamma\big/_{[a_{i-1},\, a_i]} = \gamma_i$$

[c] $f: U \longrightarrow \mathbb{C}$ continuous, Define

class C^1 path $\rightsquigarrow$

$$\int_\gamma f\,dz := \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt$$

substitute

$z = \gamma(t)$

$dz = \gamma'(t)\,dt$

This is independent of orientation preserving reparametrization

$$\int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_{\hat{a}}^{\hat{b}} f(\hat{\gamma}(s)) \cdot \hat{\gamma}'(s)\,ds$$

$t = \phi(s)$.

This is change of variables: $f(\gamma(t)) = f(\hat{\gamma}(s))$

$$\gamma'(t)\,dt = \hat{\gamma}'(s)\,ds.$$

Remark $\displaystyle\int_{-\gamma} f\,dz = -\int_{\gamma} f\,dz$ after changing orientation

Remark The definition extends to piecewise C^1 paths

$$\int_{\gamma} f\,dz = \int_{\gamma_1} f\,dz + \dots + \int_{\gamma_n} f\,dz.$$

In particular, we can define $\displaystyle\int_{\partial R} f\,dz$, R retangle.

Remark Conway works with rectifiable paths.

In the elementary theory of analytic functions it is seldom necessary to consider arcs which are rectifiable, but not piecewise differentiable. However, the notion of rectifiable arc is one that every mathematician should know.

Ahlfors – Complex Analysis, 3rd edition

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Fundamental estimate Assume $|f| \leq M$ along γ

$$\Rightarrow \left| \int_\gamma f \, dz \right| \leq \text{length}(\gamma) \cdot M.$$

Proof $\left| \int_\gamma f \, dz \right| = \left| \int_a^b f(\gamma(t)) \cdot \gamma'(t) \, dt \right|$

$$\leq M \int_a^b |\gamma'(t)| \, dt$$

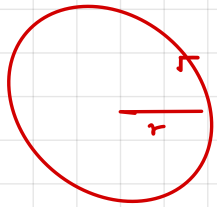
$$= M \cdot \text{length}(\gamma)$$

Augustin-Louis Cauchy (1789 - 1857) was a French mathematician who made contributions to several branches of mathematics including complex analysis.

Cauchy was a prolific writer: 800 research articles and 5 textbooks.

counterclock wise

Example A $\quad \gamma$ = circle of radius r, $\quad \gamma(t) = re^{it}$.



$z = re^{it}$

$$\int_\gamma z^n \, dz = \int_0^{2\pi} r^n e^{int} \cdot re^{it} i \, dt$$

$$= \int_0^{2\pi} r^{n+1} e^{i(n+1)t} i \, dt$$

$$= r^{n+1} \cdot \frac{e^{i(n+1)t}}{i(n+1)} \cdot i \Bigg|_{t=0}^{t=2\pi} = 0. \quad , n \neq -1.$$

When $n = -1$

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{re^{it} i}{re^{it}} \, dt = \int_0^{2\pi} i \, dt = 2\pi i$$

Remember this example!

Example B $\quad f$ admits primitive F, $\quad f = F'$.

$$\int_\gamma f \, dz = \int_a^b F'(\gamma(t)) \cdot \gamma'(t) \, dt$$

$$= \int_a^b \big(F(\gamma(t))\big)' \, dt = F(\gamma(b)) - F(\gamma(a)).$$

Path independence!

II. Existence of primitives

$U \subseteq \mathbb{C}$ open connected, f continuous. We show three results.

Proposition A TFAE

[i] f admits a primitive

[ii] $\int_\gamma f\, dz = 0$ $\forall \gamma$ piecewise C^1 loop.

Remark [i] $\Rightarrow$ [ii] is clear by Example B.

Remark $\frac{1}{z}$ doesn't admit a primitive in $U = \mathbb{C}^\times$.

since $\int_\gamma \frac{dz}{z} = 2\pi i$ by Example A.

$\Rightarrow$ $\nexists$ no logarithm in $U = \mathbb{C}^\times$.

Proposition B If $U = \Delta$ = disc. TFAE

[i] f admits primitive

[ii] $\int_{\partial R} f\, dz = 0$ for all rectangles $\overline{R} \subseteq U$.

Compare:

Prop. A	Prop B.
$U \subseteq \mathbb{C}$	$U = \Delta$
γ piecewise C^1	$\gamma = \partial R$

Proposition C If $f: U \to \mathbb{C}$ holomorphic $\Rightarrow \displaystyle\int_{\partial R} f \, dz = 0$ for all rectangles $\bar{R} \subseteq U$. (Goursat's lemma)

Remark f' is not assumed continuous. If f' is continuous an easier proof can be given.

Corollary If $f: \Delta \to \mathbb{C}$ is holomorphic,

$$\int_\gamma f \, dz = 0 \quad \forall \gamma \text{ piecewise } C^1\text{-loop in } \Delta.$$

This is a form of Cauchy's theorem.

Proof By B + C, f admits a primitive. The conclusion follows from A.

Edouard Goursat

1858 – 1936

J'ai reconnu depuis longtemps que la demonstration du theoreme de Cauchy, que j'ai donnee en 1883, ne supposait pas la continuite de la derivee.

(I have recognized for a long time that the demonstration of Cauchy's theorem which I gave in 1883 didn't really presuppose the continuity of the derivative.)

Sur la definition generale des functions analytiques, d’apres Cauchy. Trans. AMS, 1900, 14-46.

Proposition A TFAE

[i] f admits a primitive

[ii] $\int_\gamma f\, dz = 0$ $\forall \gamma$ piecewise C^1 loop.

Proof [i] $\Rightarrow$ [ii] follows by path independence.

[ii] $\Rightarrow$ [i] Fix $p \in U$ and define

$$F(q) = \int_\gamma f\, dz$$ where γ is a piecewise C^1 path in U joining p to q.

This is well-defined $\iff \int_{\gamma_1} f\, dz = \int_{\gamma_2} f\, dz$.

$\iff \int_\eta f\, dz = 0$ where $\eta = \gamma_1 + (-\gamma_2)$

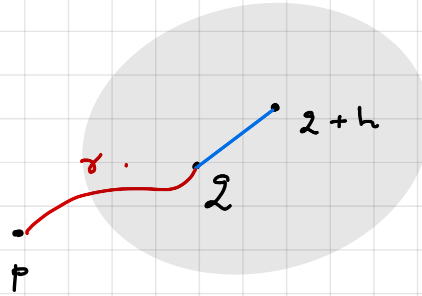
which holds due to [i].

Claim $F' = f$

Proof Fix $q \in U$, $\varepsilon > 0$. Let $\delta > 0$ with

(*) $|f(z) - f(q)| < \varepsilon$ if $z \in \Delta(q, \delta)$.

We compute



$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_z^{z+h} f(\zeta)\,d\zeta - f(z)\right|$$

$$= \frac{1}{|h|}\left|\int_z^{z+h} \big(f(\zeta)-f(z)\big)\,d\zeta\right|$$

$< \varepsilon$ by $(*)$ if $|h| < \delta$.

$$\leq \frac{1}{|h|} \cdot \text{length}\,[z, z+h] \cdot \varepsilon$$

$$= \frac{1}{|h|} \cdot |h| \cdot \varepsilon = \varepsilon \quad \Rightarrow \quad F' = f$$

Question Why can we always find a piecewise C^1 path?

Let

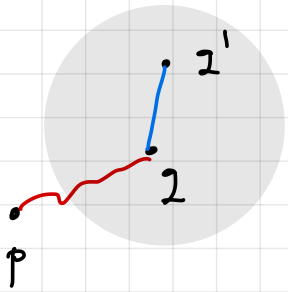
$$\mathcal{X} = \{ z \in U : \exists \text{ piecewise } C^1 \text{ path from } p \text{ to } z \}.$$

$\mathcal{X} \neq \emptyset$ since $p \in \mathcal{X}$.

We show $\mathcal{X}$ is open & closed in $\mathcal{U}$ = connected, hence

$\mathcal{X} = \mathcal{U}$, as claimed.

$\mathcal{X}$ open. Let $q \in \mathcal{X} \Rightarrow \exists R > 0$ with $\Delta(q, R) \subseteq \mathcal{U}$.



For $q' \in \Delta(q, R)$, join p to q (since $q \in \mathcal{X}$)

$\Delta(q,R)$.

join q to q' (via line segment).

$\Rightarrow q' \in \mathcal{X} \Rightarrow \Delta(q, R) \subseteq \mathcal{X} \Rightarrow \mathcal{X}$ open.

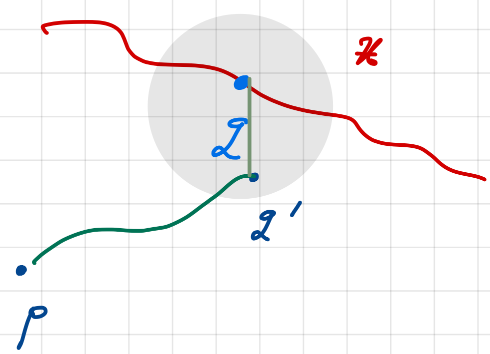
$\mathcal{X}$ closed. Let $q \in \partial \mathcal{X}$. We show $q \in \mathcal{X}$.

Let $q' \in \mathcal{X}$, $q' \in \Delta(q, \varepsilon) \subseteq \mathcal{U}$.

Join p to q' by a piecewise C^1 path. &

q' to q by line segment thus joining p to q

$\Rightarrow q \in \mathcal{X}$.

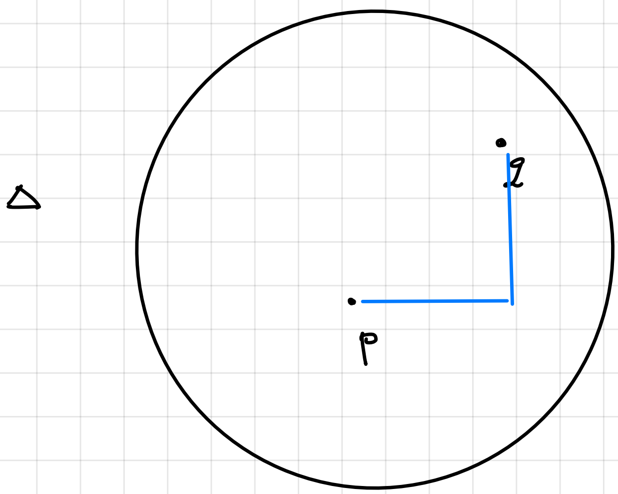


Proposition B If $U = \Delta =$ disc. TFAE

$\boxed{i}$ f admits primitive

$\boxed{ii}$ $\displaystyle\int_{\partial R} f\,dz = 0$ for all rectangles $\overline{R} \subseteq U$.

Proof We only need $\boxed{ii} \Rightarrow \boxed{i}$. Let $p \in U$.

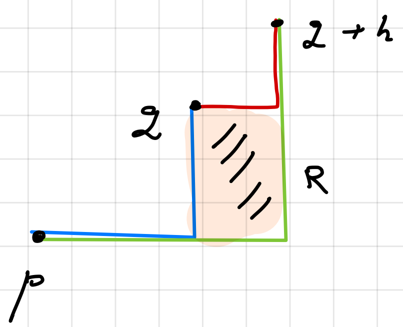


Define $F(q) = \displaystyle\int_\gamma f\,dz$ where

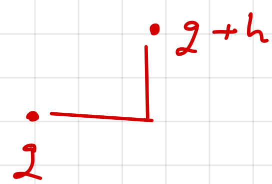
γ is a path from p to q. consisting of two segments parallel to the axes. Such a path exists since $U = \Delta =$ disc.

Claim $F' = f$.

Proof $$F(q+h) - F(q) = \int_p^{q+h} f\,dz - \int_p^{q} f\,dz = \int_q^{q+h} f\,dz$$



because $\displaystyle\int_{\partial R} f\,dz = 0.$



For the red path from q to $q+h$, the same argument applies, the length of the path $\leq 2|h|$.