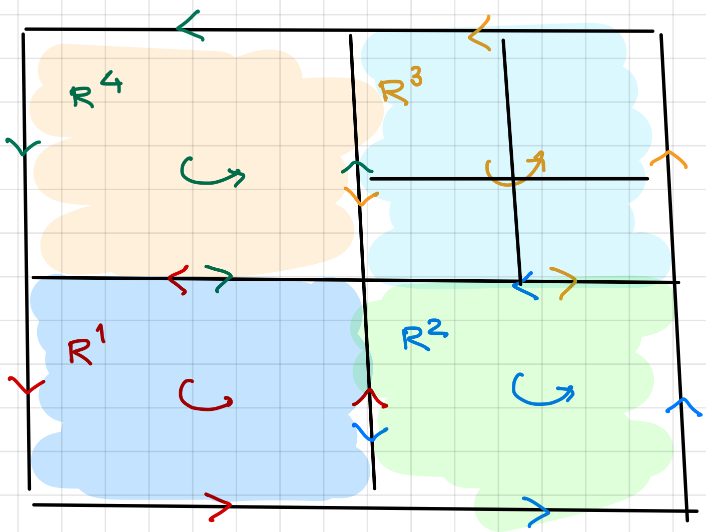
Math 220 A – Lecture 5

October 16, 2023

Proposition C If $f: U \to \mathbb{C}$ holomorphic $\Rightarrow \int_{\partial R} f\, dz = 0$

for all rectangles $\bar{R} \subseteq U$. (Goursat's lemma)

(We tacitly assume rectangles have sides parallel to axes.)



Proof Let $A = \left| \int_{\partial R} f\, dz \right|$.

Let $\varepsilon > 0$ arbitrary. Wish $A = 0$. We will show

$$A < K\varepsilon \quad \forall \varepsilon > 0.$$

for some $K > 0$.

Subdivide rectangle R into 4 equal rectangles R^1, R^2, R^3, R^4.

$$\Rightarrow A = \left| \int_{\partial R} f\, dz \right| = \left| \sum_{j=1}^{4} \int_{\partial R^j} f\, dz \right| \leq \sum_{j=1}^{4} \left| \int_{\partial R^j} f\, dz \right|.$$

$\Rightarrow \exists$ rectangle (out of R^1, R^2, R^3, R^4), call it $R^{(1)}$, with

$$\frac{A}{4} \leq \left| \int_{\partial R^{(1)}} f\, dz \right|$$

Continue inductively. We obtain a sequence of rectangles

$$R \supseteq R^{(1)} \supseteq R^{(2)} \supseteq \dots \quad , \operatorname{diam} R^{(n)} \to 0.$$

such that
$$\frac{A}{4^n} \le \left| \int_{\partial R^{(n)}} f \, dz \right|.$$

By compactness, $\bigcap_{n=0}^{\infty} R^{(n)} = \{c\}$. Since f is holomorphic

$$\left| \underbrace{\frac{f(z) - f(c)}{z - c}}_{\chi(z)} - f'(c) \right| < \varepsilon \quad \text{if } z \in \Delta(c, \delta), \text{ for some } \delta > 0.$$

$$\Rightarrow \quad |\chi(z)| < \varepsilon \quad \& \quad f(z) = f(c) + (z-c) f'(c) + (z-c)\chi(z).$$

$$\Rightarrow \frac{A}{4^n} \le \left| \int_{\partial R^{(n)}} f \, dz \right| = \left| \int_{\partial R^{(n)}} \underbrace{f(c) + (z-c) f'(c)}_{0 \text{ admits primitive}} + (z-c)\chi(z) \, dz \right|$$

$$= \left| \int_{\partial R^{(n)}} (z-c)\,\chi(z)\, dz \right|. \qquad R^{(n)} \subseteq \Delta(c, \delta) \text{ if } n \gg 0.$$

$$\le \operatorname{diam}(R^{(n)}) \cdot \varepsilon \cdot \operatorname{length}(\partial R^{(n)}).$$

$$= \varepsilon \cdot \frac{\operatorname{diam}(R)}{2^n} \cdot \frac{\operatorname{length}(\partial R)}{2^n} = \frac{\varepsilon}{4^n} K.$$

$$\Rightarrow \quad A < K\varepsilon \quad \forall \varepsilon > 0 \quad \Rightarrow \quad A = 0.$$

HWK 3, # 5

Remark A simpler proof can be given using Green's theorem if f' is assumed continuous. The point is that we don't make this assumption.

Remark In IV.8, Conway uses triangles versus rectangles.

Corollary $f : \Delta \longrightarrow \mathbb{C}$ holomorphic

$$\overset{C}{\Longrightarrow} \int_{\partial R} f\, dz = 0 \quad \forall\ \bar{R} \subseteq \Delta$$

$$\overset{B}{\Longrightarrow} f \text{ admits a primitive}$$

$$\overset{A}{\Longrightarrow} \int_{\gamma} f\, dz = 0 \quad \forall\ \gamma \text{ piecewise } C^1 \text{ loop in } \Delta$$

We seek improvements

New assumption.

(*) $f : U \longrightarrow \mathbb{C}$ continuous, f holomorphic in $U \setminus \{a\}$.

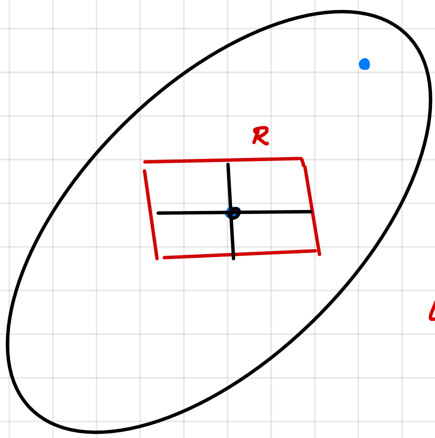
Proposition C^+ f satisfies $(*)$ then $\int_{\partial R} f\,dz = 0$.

for all $\bar{R} \subseteq U$.

Proof $\boxed{i}$ If a is outside $\bar{R}$, let $U^{new} = U \setminus \{a\}$

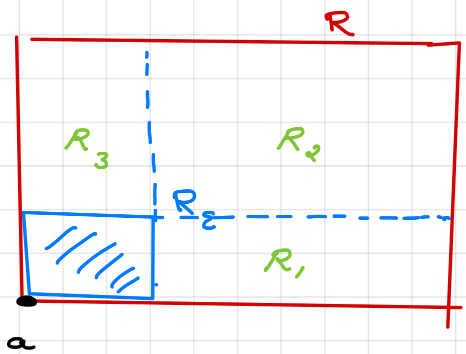
& apply Proposition C to (f, U^{new})

$$\Longrightarrow \int_{\partial R} f\,dz = 0$$



$\boxed{ii}$ If $a \in \bar{R}$, after subdividing $\bar{R}$

we may assume a is a vertex.



$\boxed{iii}$ If a is a vertex, let R_ε be a

square of side ε with vertex a.

By Proposition C we know $\int_{\partial R_j} f\,dz = 0$ for $j = 1, 2, 3$.

From here, it immediately follows $\int_{\partial R} f\,dz = \int_{\partial R_\varepsilon} f\,dz$.

To conclude, suffices $\int_{\partial R_\varepsilon} f\, dz \longrightarrow 0$ as $\varepsilon \to 0$. $(+)$

To show this, use that f is continuous at a. Then

$$|f(z)| \le |f(a)| + 1 \quad \text{if } z \in R_\varepsilon \text{ for } \varepsilon \text{ small}$$

$$\Rightarrow \left| \int_{\partial R_\varepsilon} f\, dz \right| \le (|f(a)| + 1)\, \underbrace{\text{length}(\partial R_\varepsilon)}_{4\varepsilon} \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

Corollary$^+$ $f : \Delta \longrightarrow \mathbb{C}$ continuous, f holomorphic in $\Delta \setminus \{a\}$

$$\overset{C^+}{\Longrightarrow} \int_{\partial R} f\, dz = 0 \quad \forall\, \bar{R} \subseteq \Delta$$

$\overset{B}{\Longrightarrow}$ f admits a primitive

$$\overset{A}{\Longrightarrow} \int_\gamma f\, dz = 0 \quad \forall\, \gamma \text{ piecewise } C^1 \text{ loop in } \Delta$$

Cauchy Integral Formula (local form)

$f: \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic. Let $\overline{\Delta} \subseteq \mathcal{U}$, $a \in \Delta$. Then

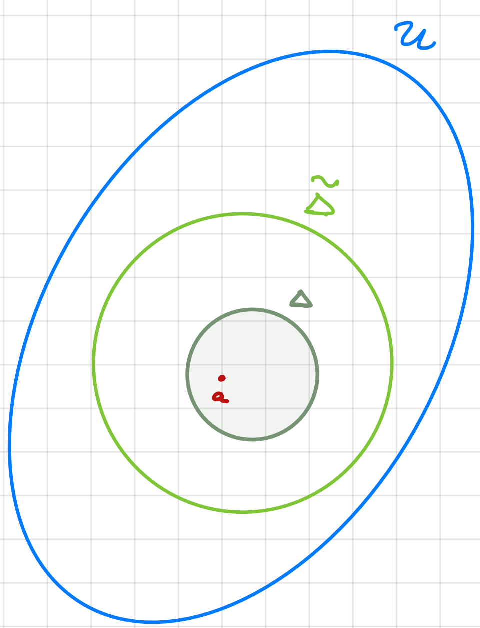
disc

$$\boxed{f(a) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(z)}{z-a}\, dz}$$

positively oriented (counterclockwise)

Remark The formula shows $f|_{\partial\Delta}$ determines f in Δ!

Proof Let



$$F(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & \text{if } z \neq a \\ f'(a) & \text{if } z = a \end{cases}$$

$\Rightarrow$ F continuous on $\mathcal{U}$. & holomorphic in $\mathcal{U} \setminus \{a\}$.

Let $\tilde{\Delta}$ s.t. $\overline{\Delta} \subseteq \tilde{\Delta} \subseteq \overline{\tilde{\Delta}} \subseteq \mathcal{U}$.

Apply Corollary^{+} to $F/_{\tilde{\Delta}}$ and $\gamma = \partial\Delta$. We find

$$\int_{\partial\Delta} F\,dz = 0 \implies \int_{\partial\Delta} \frac{f(z) - f(a)}{z-a}\,dz = 0.$$

$$\implies \frac{1}{2\pi i}\int_{\partial\Delta} \frac{f(z)}{z-a}\,dz = f(a)\cdot \underbrace{\frac{1}{2\pi i}\int_{\partial\Delta} \frac{dz}{z-a}}_{1.\ \text{(next lemma)}} = f(a)$$

$\implies$ Local Cauchy.

Remark This is a version of Conway IV.2.6.

The difference with Conway is that we do not assume continuity of the derivative! This assumption is removed in Conway later in IV.8, so we are arriving at the same conclusions in the end.

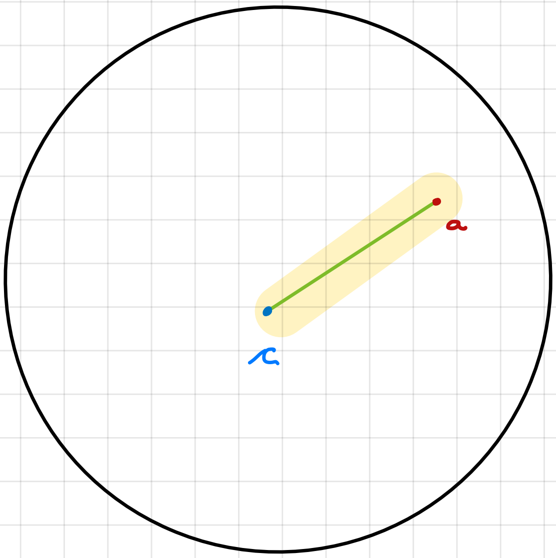
The presentation here is closer to Ahlfors, Chapter IV.

Lemma If $a \in \Delta \Longrightarrow \displaystyle\int_{\partial\Delta} \frac{dz}{z-a} = 2\pi i$

$\partial\Delta \rightsquigarrow$ positively oriented

Proof

Step 1 Let c be the center of Δ. Then



$$\int_{\partial\Delta} \frac{dz}{\underbrace{z-c}_{w}} = 2\pi i$$

$$\Longleftrightarrow \int_{\partial\Delta(0,R)} \frac{dw}{w} = 2\pi i$$

which we saw in Lecture 4.

Step 2

It suffices to show $\displaystyle\int_{\partial\Delta} \left(\frac{dz}{z-a} - \frac{dz}{z-c} \right) = 0 \Longleftrightarrow \int_{\partial\Delta} h \, dz = 0$.

Let $h(z) = \dfrac{1}{z-a} - \dfrac{1}{z-c}$. We show that h admits a primitive in $\mathbb{C} \setminus [a,c]$. Let

principal branch

$$\operatorname{Log} \frac{z-a}{z-c} = g(z)$$

check

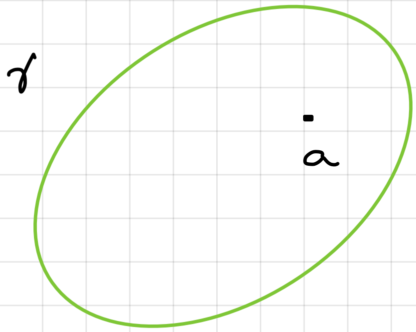
$\Longrightarrow g' = h$. $\Longrightarrow \displaystyle\int_{\partial\Delta} h \, dz = 0$ by Proposition A.

Issue We need to show $\dfrac{z-a}{z-c} \in \mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

If $\dfrac{z-a}{z-c} = -u$, $u \in \mathbb{R}_{\geq 0}$ $\Longrightarrow z = a \cdot \dfrac{1}{u+1} + c \cdot \dfrac{u}{u+1}$. $\in$ segment from a to c., false!

Winding number (index)

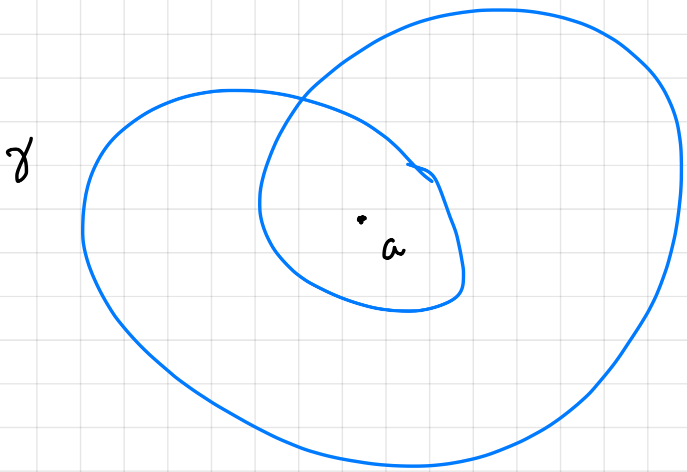
Let $a \notin \{\gamma\}$ (= Image of γ). loop



Define
$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$$

Conway IV.4

Example A



γ circle positive orientation

$\Longrightarrow$ $n(\gamma, a) = 1$ if $a \in \operatorname{Int}\gamma$.

by the Lemma.

goes k times around 0.

Example B

$\gamma_k(t) = e^{2\pi i t k}$, $0 \le t \le 1$.

$\Longrightarrow$ $n(\gamma_k, 0) = k$. Indeed

$$n(\gamma_k, 0) = \frac{1}{2\pi i}\int_{\gamma_k} \frac{dz}{z} =$$

$$= \frac{1}{\cancel{2\pi i}}\int_0^1 \frac{\cancel{e^{2\pi i t k}} \cdot \cancel{2\pi i} k}{\cancel{e^{2\pi i t k}}}\, dt$$

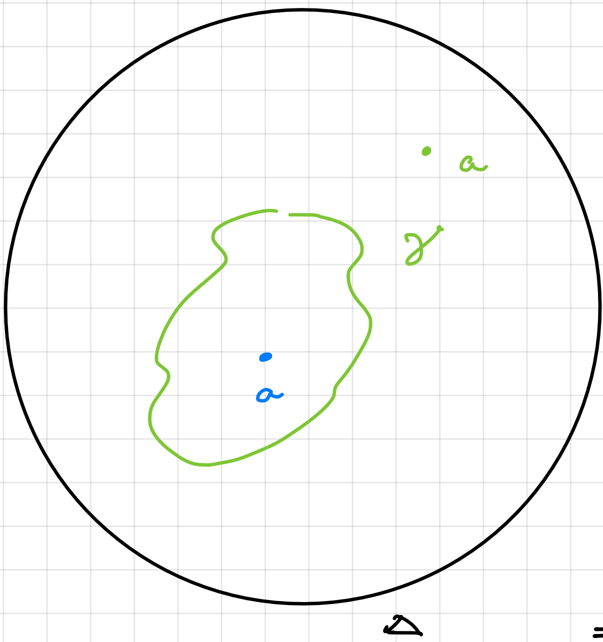
$= k$.

Stronger Cauchy's Integral Formula for discs

Let $f : \Delta \longrightarrow \mathbb{C}$ holomorphic.

γ closed C^1 loop in Δ, $a \in \Delta \setminus \{\gamma\}$. Then

$$f(a) \cdot n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz$$



Same proof as before!

$$F(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & \text{if } z \neq a \\ f'(a) & \text{if } z = a \end{cases}$$

Corollary^{+} shows $\int_\gamma F\, dz = 0$

$$\Rightarrow \int_\gamma \frac{f(z)-f(a)}{z-a}\, dz = 0$$

$$\Rightarrow \int_\gamma \frac{f(z)}{z-a} = f(a) \int \frac{dz}{z-a} = 2\pi i \; n(\gamma, a)\, f(a)$$

More on winding numbers.

Lemma $n(\gamma, a) \in \mathbb{Z}$ for all $a \notin \{\gamma\}$. ↙ Conway IV.4.1

Proof next time.

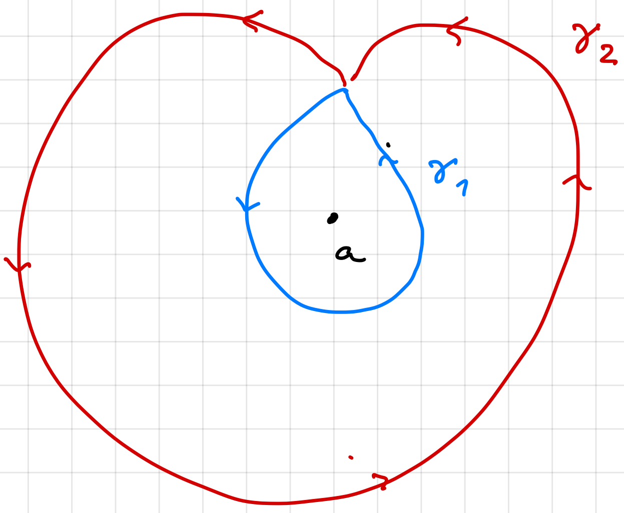
Properties

$\boxed{\text{I}}$ $n(-\gamma, a) = -n(\gamma, a)$ (change of orientation)

Proof:

$$\int_{-\gamma} \frac{dz}{z-a} = -\int_{\gamma} \frac{dz}{z-a}$$

$\boxed{\text{II}}$ $\gamma = \gamma_1 + \gamma_2 \Rightarrow n(\gamma, a) = n(\gamma_1, a) + n(\gamma_2, a)$



In the picture

$n(\gamma_1, a) = 1$, $n(\gamma_2, a) = 1$

$\Rightarrow n(\gamma, a) = 2$

Proof:

$$\int_{\gamma} \frac{dz}{z-a} = \int_{\gamma_1} \frac{dz}{z-a} + \int_{\gamma_2} \frac{dz}{z-a}.$$

$\boxed{III}$ $n(\gamma, -) : \mathbb{C} \setminus \{\gamma\} \longrightarrow \mathbb{Z}$ is locally constant

$n(\gamma, a) = 0$ for a in the unbounded component of $\mathbb{C} \setminus \{\gamma\}$.

Proof next time.