Math 220A – Lecture 6

October 18, 2023

Last time : Winding number (index)

- γ piecewise C^1 loop, $a \notin \{\gamma\}$.

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} \in \mathbb{Z}$$

Lemma $n(\gamma, a) \in \mathbb{Z}$ for all $a \notin \{\gamma\}$. ← Conway IV.4.1

Proof $n(\gamma, a) = \frac{1}{2\pi i} \int_\alpha^\beta \frac{\gamma'(s)}{\gamma(s)-a}\, ds$ where

$\gamma : [\alpha, \beta] \longrightarrow U$ is a piecewise C^1 loop $\gamma(\alpha) = \gamma(\beta)$.

Consider $h(t) = \int_\alpha^t \frac{\gamma'(s)}{\gamma(s)-a}\, ds.$, $h(\alpha) = 0$.

Want $h(\beta) \in 2\pi i \mathbb{Z}$.

Compute

$$h'(t) = \frac{\gamma'(t)}{\gamma(t)-a}.$$

$$\Rightarrow \left(e^{-h(t)} \cdot (\gamma(t) - a)\right)' = e^{-h(t)}\Big(\underbrace{-h'(t)(\gamma(t)-a) + \gamma'(t)}_{0}\Big)$$

$$\Rightarrow e^{-h(t)}(\gamma(t) - a) \text{ constant.}$$

Let $t=\alpha$, $t=\beta$. Then, we find

$$\underbrace{e^{-h(\alpha)}}_{1}\,(\gamma(\alpha)-a) = e^{-h(\beta)}(\gamma(\beta)-a).$$

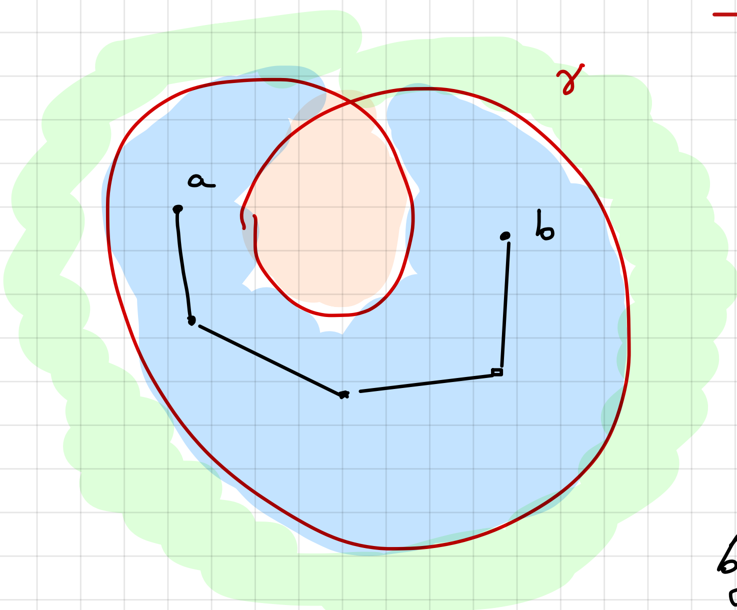
same, γ loop

$$\Rightarrow e^{-h(\beta)} = 1 \Rightarrow h(\beta) \in 2\pi i\,\mathbb{Z}.$$ QED.

<u>Lemma</u> $n(\gamma,-): \mathbb{C}\setminus\{\gamma\} \longrightarrow \mathbb{Z}$ is locally constant and $n(\gamma,a)=0$ for a in the unbounded component of $\mathbb{C}\setminus\{\gamma\}$.

$\rightsquigarrow$ Conway IV. 4.4.

<u>Proof</u>



Let R be a component of $\mathbb{C}\setminus\{\gamma\}$. If $a, b \in \bar{R}$

$\Rightarrow$ a, b can be joined by a polygonal path in R.

This is the same argument used in Lecture 4 to show

we can join by piecewise C^1 path.

Suffices to show if $\overline{ab} \subseteq R$

→ segment from a to b

$$\Rightarrow \quad n(\gamma, a) = n(\gamma, b)$$

$$\Longleftrightarrow \quad \int_\gamma dz \left(\frac{1}{z-a} - \frac{1}{z-b} \right) = 0.$$

This is true since $\operatorname{Log} \dfrac{z-a}{z-b}$ is a primitive of the

integrand. & Proposition A. We showed $\operatorname{Log} \dfrac{z-a}{z-b}$ is well

defined in $\mathbb{C} \setminus \overline{ab}. \supseteq \{\gamma\}$ in Lecture 5.

If $\mathcal{U}$ is the unbounded component, let

$R >> 0.$ such that $\{\gamma\} \subseteq \Delta(0, R)$. Let m be

the value of $n(\gamma, \_)$ on $\mathcal{U}$. Want $m = 0$.

Pick $|a| \geq 2R$, $a \in U$. Then

$$|z-a| \geq |a|-|z| \geq 2R - R = R \text{ if}$$

$z \in \{\gamma\}$ $\Rightarrow$

$$|m| = |n(\gamma, a)| = \frac{1}{2\pi}\left|\int_{\gamma} \frac{dz}{z-a}\right| \leq$$

$$\leq \frac{1}{2\pi} \cdot \frac{1}{R} \cdot \text{length}(\gamma).$$

Make $R \longrightarrow \infty$ $\Rightarrow$ $n(\gamma, a) = m = 0$.

Rudiments of algebraic topology

$$\pi_1(X) = (\text{based}) \text{ loops in } X /\sim$$

homotopy

can be shown isomorphism

$$\pi_1(\mathbb{C} \setminus \{a\}) \longrightarrow \mathbb{Z}$$ well-defined by [b] below

$$\gamma \longrightarrow n(\gamma, a).$$

Two questions arise

[a] Can we define integrals over γ continuous?

[b] $\gamma_1 \sim \gamma_2 \overset{?}{\Longrightarrow} n(\gamma_1, a) = n(\gamma_2, a).$

Answer to [a] YES. If f holomorphic, γ continuous we define $\int_\gamma f\,dz$. ideas from analytic continuation

We will not pursue this here.

Answer to [b] YES. Cauchy's Theorem (Homotopy)

Conway $\underline{\overline{IV}}$. 6.

We reparametrize so that the domain is $I = [0,1]$.

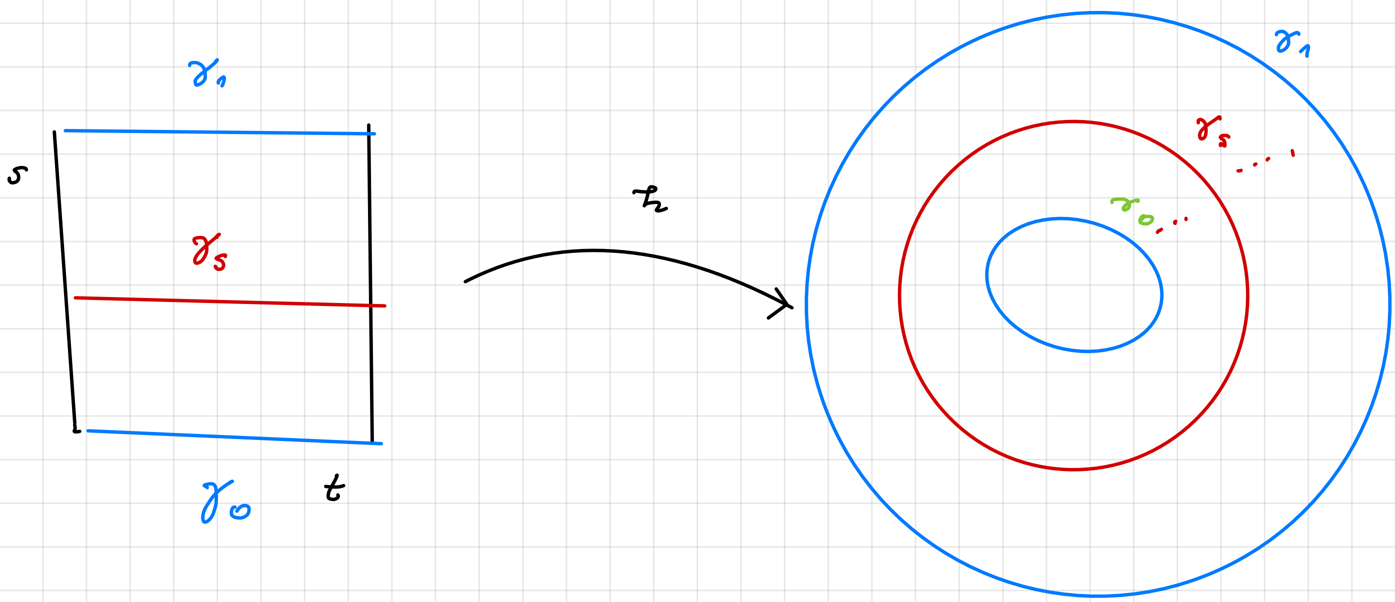
Homotopy $\gamma_0, \gamma_1 : I \longrightarrow U$ continuous loops

$\gamma_0 \overset{U}{\sim} \gamma_1$ if $\exists\, h : I \times I \longrightarrow U$ continuous

$$h(t,0) = \gamma_0(t), \quad h(t,1) = \gamma_1(t).$$

$$h(0,s) = h(1,s).$$

$\Longrightarrow \quad \gamma_s(t) = h(t,s)$. continuous loop.

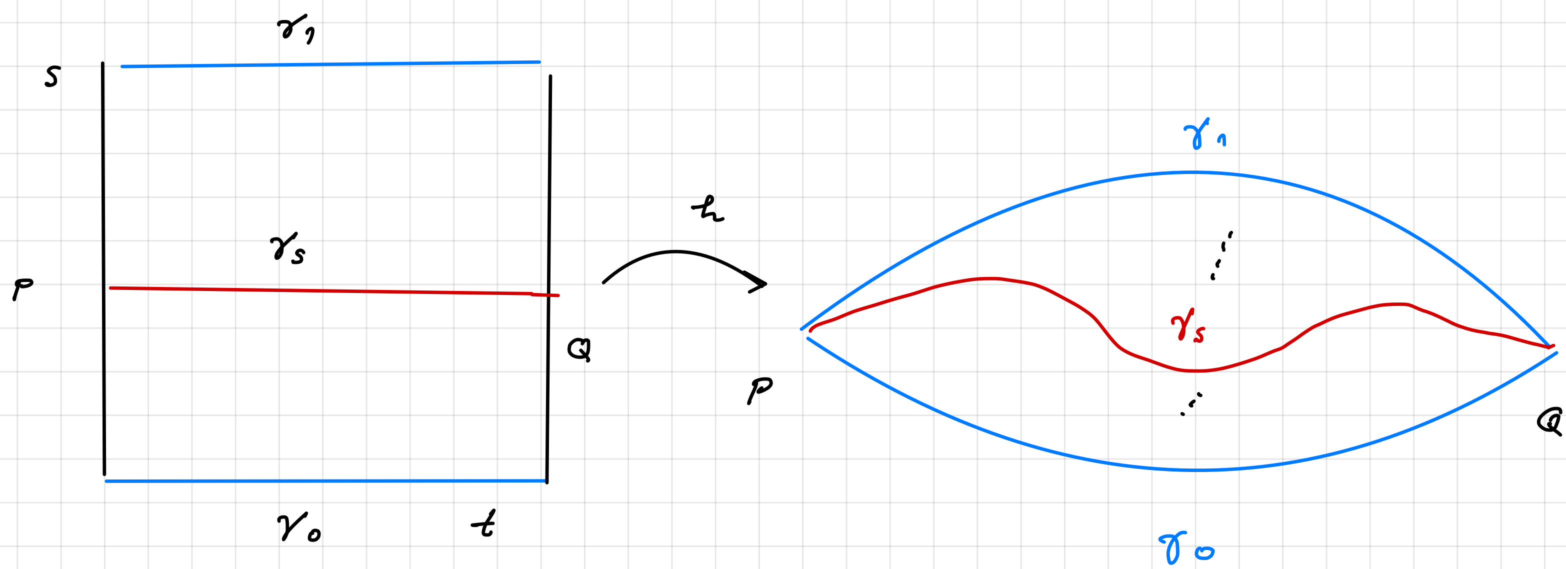


Def $\gamma_0, \gamma_1 : I \longrightarrow \mathcal{U}$ continuous paths from P to Q

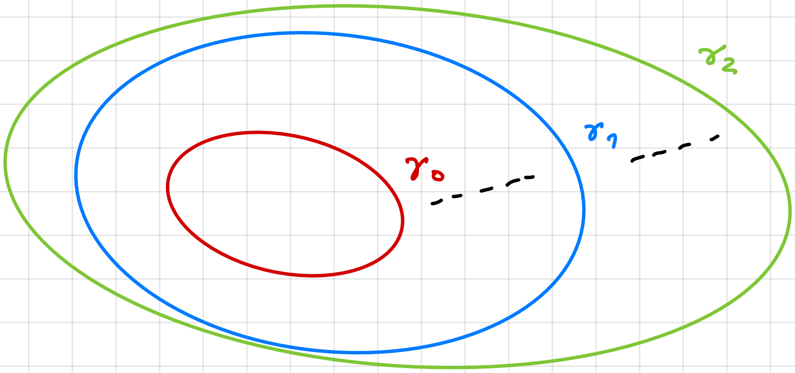
$\gamma_0 \overset{\mathcal{U}}{\underset{FEP}{\sim}} \gamma_1$ if $\exists\, h : I \times I \longrightarrow \mathcal{U}$ continuous

$$h(t,0) = \gamma_0(t) \quad , \quad h(t,1) = \gamma_1(t)$$

$$h(0,s) = P \quad , \quad h(1,s) = Q .$$



Remark $\boxed{a}$ $\sim$ is an equivalence relation.



$\gamma_0 \overset{\mathcal{U}}{\sim} \gamma_1$, $\gamma_1 \overset{\mathcal{U}}{\sim} \gamma_2 \Longrightarrow$

$\Longrightarrow \gamma_0 \overset{\mathcal{U}}{\sim} \gamma_2$

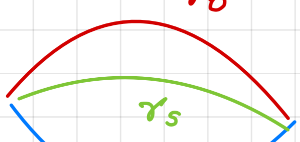
$\boxed{b}$ Check $\gamma + (-\gamma) \overset{\mathcal{U}}{\sim} 0$. $\forall \gamma$ path in $\mathcal{U}$

constant loop

$\boxed{c}$ If $\gamma_0 \overset{FEP}{\sim} \gamma_1$, let $\gamma = \gamma_0 + (-\gamma_1)$ loop

$\Longrightarrow \gamma \overset{\mathcal{U}}{\sim} 0$. as loops. Indeed let

$\Gamma_s = \gamma_s + (-\gamma_1)$.



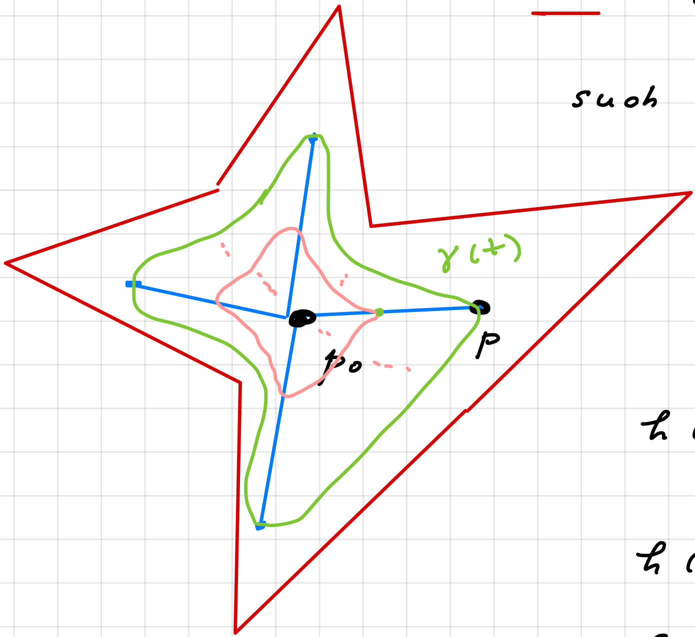
$\Gamma_0 = \gamma$. By $\boxed{b}$, $\Gamma_1 \sim 0$.

By $\boxed{a}$ $\Longrightarrow \gamma \overset{\mathcal{U}}{\sim} 0$.

Def $\mathcal{U}$ is *simply connected* if $\forall \gamma$ loop in $\mathcal{U}$,

$\gamma \overset{\mathcal{U}}{\sim} 0 \iff \pi_1(\mathcal{U}) = 0$.

Example $\mathcal{U}$ is star convex $\Rightarrow$ $\mathcal{U}$ simply connected



Def $\mathcal{U}$ star convex if $\exists\, p_0 \in \mathcal{U}$ such that $\forall\, p \in \mathcal{U} \Rightarrow \overline{p_0 p} \subseteq \mathcal{U}$.

Let γ be a loop in $\mathcal{U}$.

$$h(t,s) = s\, p_0 + (1-s)\,\gamma(t) \subseteq \mathcal{U}$$

$$h(t,0) = \gamma(t)$$

$$h(t,1) = p_0 \qquad \Rightarrow \gamma \sim 0.$$

IV. 6. 7

Cauchy's Theorem (Homotopy version)

$f: U \longrightarrow \mathbb{C}$ holomorphic, $\gamma_0 \overset{U}{\sim} \gamma_1$ piecewise C^1 loops in U $\Rightarrow \int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$.

Remarks $\boxed{i}$ $\gamma \overset{U}{\sim} 0 \Rightarrow \int_{\gamma} f\,dz = \int_{0} f\,dz = 0.$

If U simply connected $\Rightarrow \int_{\gamma} f\,dz = 0 \;\; \forall \gamma \; C^1$ loop in U.

$\boxed{ii}$ γ_1, γ_2 piecewise C^1 paths, $\gamma_1 \overset{FEP}{\sim} \gamma_2$

$\Rightarrow \int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz$. Indeed let $\gamma = \gamma_1 + (-\gamma_2)$.

By $\boxed{i}$ $\Rightarrow \int_{\gamma} f\,dz = 0 \Rightarrow \int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz.$

Conway IV. 6. 13

$\boxed{iii}$ $\gamma_0 \overset{U}{\sim} \gamma_1$, $U \subseteq \mathbb{C} \setminus \{a\}$ piecewise C^1 loops in $U \subseteq \mathbb{C} \setminus \{a\}$ $\Rightarrow \int_{\gamma_0} \frac{dz}{z-a} = \int_{\gamma_1} \frac{dz}{z-a}$

$\Rightarrow n(\gamma_0, a) = n(\gamma_1, a).$

This proves a previous assertion.

Remark The homotopy in Cauchy's theorem is not assumed to be C^1.

Existence of primitives in simply connected sets

If $\mathcal{U}$ simply connected, $f: \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic

$$\Rightarrow \int_\gamma f\,dz = 0. \text{ by Remark } \boxed{1}$$

$\Rightarrow$ Prop A, f has a primitive

Corollary Any holomorphic function in a simply connected set admits a primitive. Conway IV. 6.16

Take $f(z) = \frac{1}{z}$. A primitive is a branch of logarithm.

Corollary Let $\mathcal{U} \subseteq \mathbb{C} \setminus \{0\}$ simply connected. We can define a branch of logarithm in $\mathcal{U}$.

Conway IV. 6.17

Cauchy's Theorem (Homotopy version)

$f : U \longrightarrow \mathbb{C}$ holomorphic, $\gamma_0 \overset{U}{\sim} \gamma_1$ piecewise C^1 loops in U $\Longrightarrow \int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$.

Remark We prove a seemingly stronger result

Next time

Cauchy's Theorem$^+$ (Homotopy version).

(+) $f : U \longrightarrow \mathbb{C}$ continuous, holomorphic in $U \setminus \{a\}$

$\Longrightarrow \int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$ if $\gamma_0 \overset{U}{\sim} \gamma_1$ are piecewise C^1 loops.

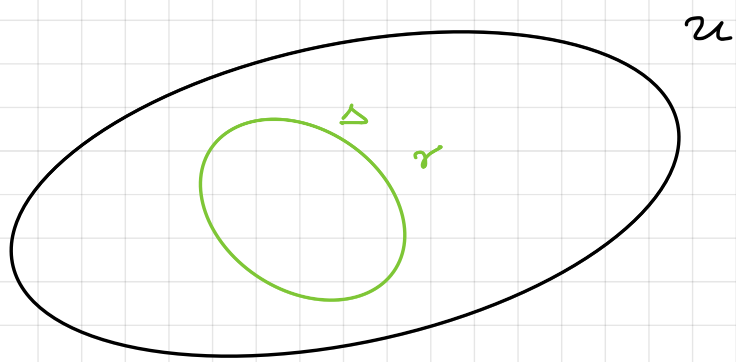
We need this stronger form to prove:

Cauchy's Integral Formula (CIF)

$f : U \longrightarrow \mathbb{C}$ holomorphic, $\gamma \overset{U}{\sim} 0$, $a \in U \setminus \{\gamma\}$

$$n(\gamma, a)\, f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a}\, dz$$

Remark This generalizes Local Cauchy's Integral Formula we proved before. In that case, $\gamma = \partial \Delta$ where $\overline{\Delta} \subseteq U$.



Proof of CIF

Let $$F(z) = \begin{cases} \dfrac{f(z) - f(a)}{z-a} & , z \neq a \\ f'(a) & , z = a \end{cases}$$

$\Longrightarrow$ F continuous in U, holomorphic in $U \setminus \{a\}$.

$$\Longrightarrow \int_\gamma F \, dz = 0 \quad \text{by Cauchy}^+$$

$$\Longrightarrow \frac{1}{2\pi i} \int_\gamma \frac{f(z) - f(a)}{z-a} \, dz = 0$$

$$\Longrightarrow \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} = f(a) \cdot \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = f(a) \cdot n(\gamma, a).$$

QED.

Remark

$$\begin{array}{ccc} \text{Homotopy Cauchy}^+ & \Longrightarrow & \text{CIF} \\ \Downarrow & \swarrow & \\ \text{Homotopy Cauchy} & & \end{array}$$

In fact CIF $\Longrightarrow$ Homotopy Cauchy by using CIF

for $\gamma = \gamma_0 + (-\gamma_1)$ & the function $(z-a) f(z)$

Example $|a| < |b|$. We compute

$$\int_{|z|=r} \frac{e^z}{(z-a)(z-b)}\, dz$$

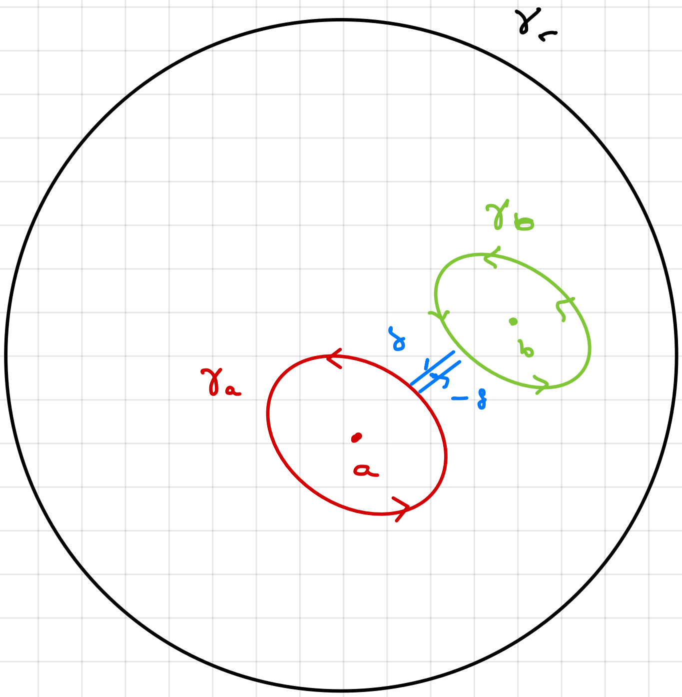
$\boxed{i}$ $r < |a|$, the integrand is holomorphic so answer $= 0$.

$\boxed{ii}$ $|a| < r < |b|$. Write

holomorphic in $|z| \le r$.

$$\int_{|z|=r} \frac{e^z/(z-b)}{z-a} = 2\pi i \cdot \left. \frac{e^z}{z-b} \right|_{z=a} = 2\pi i \cdot \frac{e^a}{a-b}.$$

$\boxed{iii}$ $|a| < |b| < r$ Let $\gamma_r = \{|z| = r\}$.



Let $f(z) = \dfrac{e^z}{(z-a)(z-b)}$.

Let γ_a, γ_b be two circles centered at a, b and δ a segment joining them.

Let $\gamma = \gamma_a + \delta + \gamma_b + (-\delta)$.

Note $\gamma \sim \gamma_r$ in $\mathbb{C} \setminus \{a, b\}$.

By homotopy Cauchy

$$\int_{\gamma_r} f\,dz = \int_{\gamma} f\,dz = \int_{\gamma_a} f\,dz + \int_{\gamma_b} f\,dz + \cancel{\int_{\delta} f\,dz} + \cancel{\int_{-\delta} f\,dz}$$

$$= \int_{\gamma_a} \frac{e^z/z-b}{z-a}\,dz + \int_{\gamma_b} \frac{e^z/z-a}{z-b}\,dz$$

$$= 2\pi i \cdot \frac{e^z}{z-b}\Big|_{z=a} + 2\pi i \frac{e^z}{z-a}\Big|_{z=b}$$

$$= 2\pi i \cdot \frac{e^a - e^b}{a-b}.$$